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CLUSTER DIMENSION OF A NETWORK

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Abstract

We define the *Cluster Dimension* of a network is the minimum cardinality of a subset *S* of the set of nodes having the property that for any two distinct nodes *x* and *y*, there exist the node s_1, s_2 (need not be distinct) in *S* such that $|d(x, s_1) - d(y, s_1)| \geq 1$ and $d(x, s_2) < d(x, s)$ for all $s \in S - \{s_2\}$. In this paper, we obtain sufficient conditions for a graph of cluster dimension *n* and a tight upper bound for the number of nodes of a network with prescribed dimension in terms of diameter. Also give an algorithm to determine cluster basis and dimensions of a given graph to show the *NP* completeness.

Keywords: Metric dimension, metric basis, cluster dimension, cluster basis

AMS Subject Classification No.: 05C78, 05C12, 05C15

1. INTRODUCTION

Every network can be viewed as a graph in which the nodes represent the processors and an edge between any two nodes indicate the connection between the processors corresponding to the nodes. In [4], navigations are studied in a graph-structured framework in which the navigating agent (the robot) moves from node to node of a graph space. The robot can locate itself by the presence of the distinct codes assigned for the nodes of the graph. There are several methods to associate a code for a node. For example, In [8], Paul F Tsuchiya showed amethod of assigning the codes to the nodes by decomposing the network into sub networks. The method of approach in [8] is random and the code associated depends only based on the number of sub divisions. However, a mathematical approach for the assignment of distinct codes given by F. Harary et al [4] and are studied by various other authors in [2,7,9-14] are purely depends on the other invariants associated with the network namely, diameter, distance between two nodes etc., The codes generated by the above methods can easily be implement to locate the node. The concept identifies each and every nodes uniquely but fails to assign a unique processor to perform the needful at that node. So, the purpose of this paper is to innovate a new methodology to generate a unique code and thereby implement it into a real network topology.

Throughout this paper we write $G(V, E)$ or simply *G*, to denote a graph on a finite non empty set *V* of nodes having its edge set *E*. All the graphs considered in this paper are simple finite undirected and connected. The terms not defined here may found in [1,3,5]. For any two nodes *u* and *v* the distance between *u* and *v*, denoted by $d(u, v)$, is the length of the shortest path between them. For a given graph *G*, there are number of properties related to distance between two nodes are widely studied by various authors.

2. CLUSTER DIMENSION

We define the *Cluster Dimension* of a network is the minimum cardinality of a subset *S* of the set of nodes having the property that for any two distinct nodes *x* and *y*, there exist the nodes s_1, s_2 (need not be distinct) in *S* such that $|d(x, s_1) - d(y, s_1)| \geq 1$ and

$d(x, s_2) < d(x, s)$ for all $s \in S - \{s_2\}$. The elements of the set *S* are called the *routers* or *route node* or (*resource locators*) and the set *S* is called a *Cluster basis*. The Cluster dimension of a graph *G* is denoted by $\beta_c(G)$.

Example: In the following figure 2.1 (a) to 2.1 (*l*), we observe that the set *S* (shown in doted lines) is a Cluster basis. However, the set *S* is not unique, in fact the set of any two adjacent nodes also serve as cluster basis. The set *S* need not contain two adjacent nodes for example for a cycle C_{10} any two non-antipodal nodes which are at an odd distances serve the same purpose.

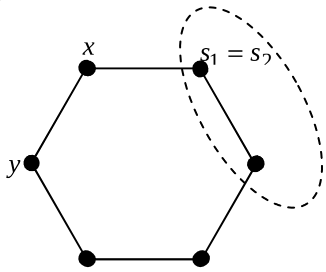


Figure 2.1 (*a*)

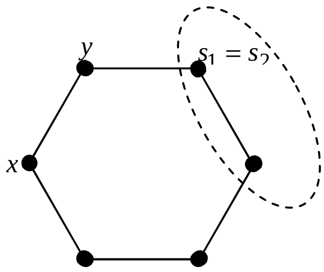
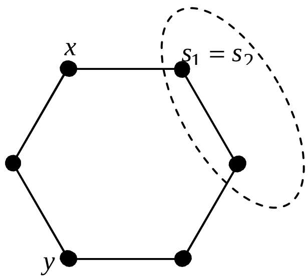


Figure 2.1 (*b*)



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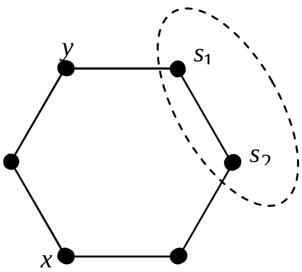

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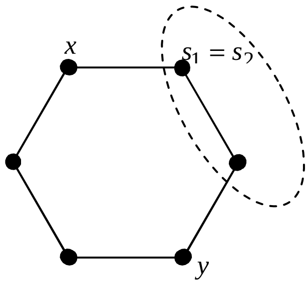
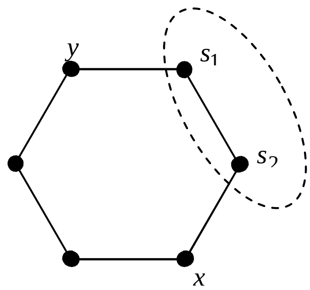


Figure 2.1 (*e*)



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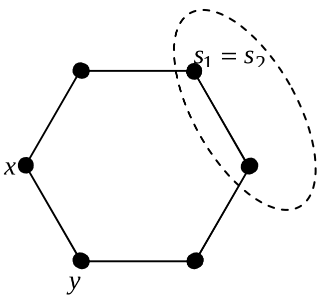

Figure 2.1 (*g*)

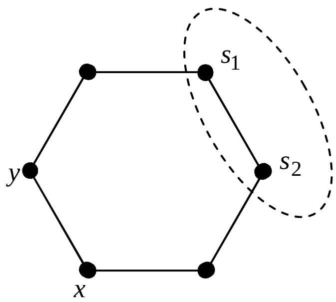

Figure 2.1 (*h*)

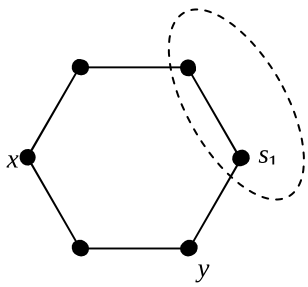

Figure 2.1 (*i*)

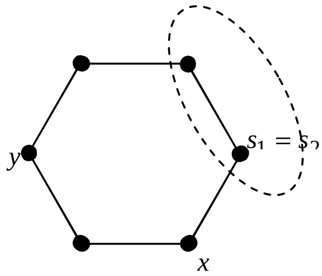

Figure 2.1 (*j*)

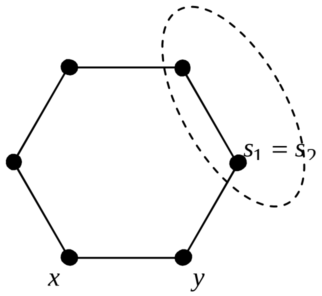

Figure 2.1 (*k*)

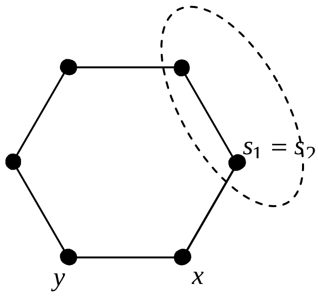

Figure 2.1 (*l*)

The invariant $\beta_c(G)$ is well defined; in fact the set V itself is a set of routers. However it need not be minimal for most of the graphs. The first constraint, namely $|d(x, s_1) - d(y, s_1)| > 0$, in the definition of $\beta_c(G)$ shows that, the metric dimension $\beta(G) \leq \beta_c(G)$. Hence, cluster basis is a super set of metric basic with an additional property. Hence, as the metric dimension of a graph *G* is one if and only if *G* is a path [4], it follows that for any graph *G* which is not a path $\beta_c(G) > 1$. For a path, the set consisting of any one of end node is a cluster basis. We state this fact as;

Theorem 2.1: $\beta_c(G) = 1$ *if and only if G is a path.*

Remark 2.2: The second constraint in the definition of $\beta_c(G)$ is required to assign a unique router for each node of the network.

Since cluster dimension of a complete graph on *n* nodes is *n* and by Theorem 2.1 we conclude that for a connected graph *G* of order *n*, $1 \leq \beta_c(G) \leq n$, for $n > 1$. We now see the complexity in computation of the cluster dimension of a graph *G*.

Consider the graphs G_1 and G_2 shown in the figures 2.2(*a*) and 2.2 (*b*).

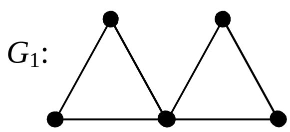

Figure 2.2 (*a*)

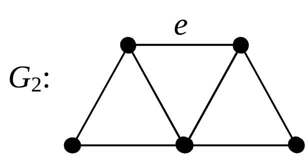

Figure 2.2 (*b*)

The cluster dimension of G_1 is 5. The graph G_2 is obtained from G_1 by adding an edge '*e*' and $\beta_c(G_2)$ = 5. Thus, addition of an edge retains the cluster dimension in this case. However it is not true in general (example: path). Further by adding an edge, even though the cluster dimension is increased or stable in the above cases, this cannot be true
for all the graphs. As a counter example we observe that for the following graph G in
figure 2.3:Now, there are graphs whose cluster dimension decreases by adding an edge, as a counter example consider following graph G in figure 2.3:

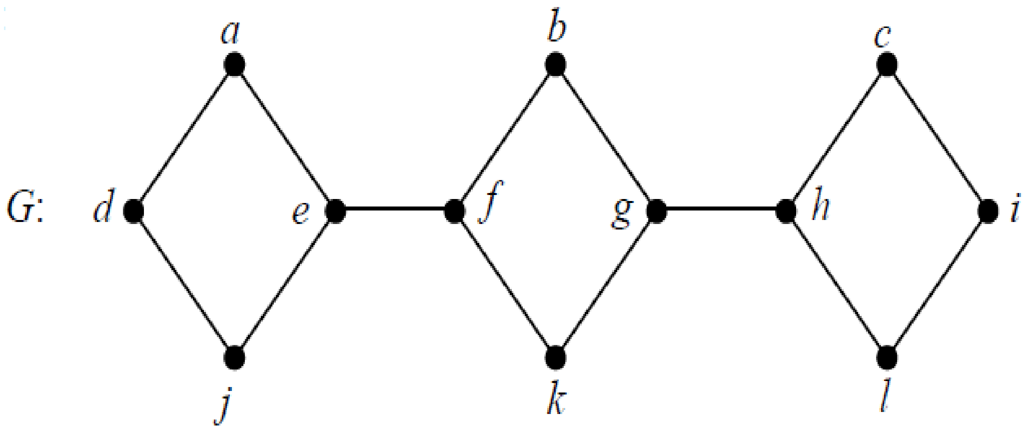

Figure 2.3

(i) $\beta_c(G)$ = 3 and a cluster basis $M = \{a, b, c\}$.

(ii) $\beta_c(G + ad)$ = 2 and a cluster basis M = {c, k}.

If S is a cluster basis of a network having node set V, then the set of nodes in $V - S$ is located uniquely by a locator $s \in S$ is called *cluster of s*, and is denoted by C(s). That is, $C(s) = \{x \in V - S \mid d(s, x) < d(v, x), \forall v \in S - \{s\}\}$. Let $C[s] = C(s) \cup \{s\}$. Then the collection $\{C(s)\}_{s \in S}$ of clusters contains exactly $\beta_c(G)$ members and they form a partition of the set of nodes of the network. Further, each cluster contains exactly one locator (routers), the one which is very close to each element in that cluster but no in any other cluster of the network.

The following is a simple observation on cluster dimension.

Lemma 2.3: *If S is a cluster basis for a graph G and $s_1, s_2 \in S$, then $v \in C[s_1]$, implies that $d(s_1, s_2) < 2\, d(s_2, v)$.*

Proof: Since $v \in C[s_1]$, we have $d(s_1, v) < d(s_2, v)$ and hence by triangular inequality, $d(s_1, s_2) \leq d(s_1, v) + d(v, s_2) < d(s_2, v) + d(v, s_2) = 2\, d(s_2, v)$. ♦

Lemma 2.4: *If a node $x \in C[s_1]$ and P be a shortest path between x and s_1, then $V(P) \subseteq C[s_1]$.*

Proof: Suppose to contrary that there exists a node *y* in a shortest path *P* between *x* and s_1, where $x \in C(s_1)$, such that $y \notin C[s_1]$. Then $y \in C[s_2]$ for some $s_2 \in S$, the cluster basis of the network, and $d(s_2, y) < d(s_1, y)$. Let $d(s_1, x) = a$, $d(s_2, y) = b$ and $d(y, x) = k$. Then, as $y \in C[s_2]$, we get $b < a - k$ and hence $d(s_2, x) < b + k < a$, which is a contradiction to the fact that $x \in C[s_1]$ (since $d(s_1, x) < d(s, x),$ for all $s \in S$). ♦

Remark 2.5: If a node *v* at a distance *k* from any router in *S*, then every node which is adjacent to *v* and at a distance *k* – 1 should be in *S* only. However, if a node which is at a distance less than or equal to *k* from the same router then that node need not be in *S*. For example, in figure 2.1 (*d*), the node *x* is at a distance 2 from the router s_2 and is routed by s_2, but *y* is routed by s_1 even though which is at equal distance from s_2.

Remark 2.6: If any node in a network is a full degree node (adjacent to every other node), then that node should be a router. Thus, it follows that every node in a complete graph is a router and hence $\beta_c(K_n) = n$. However, the converse of this statement need not be true. This fact is established in the next section.

3. GRAPHS OF CLUSTER DIMENSIONS *N*.

Throughout this section *n* denotes the number of nodes in the graph unless otherwise mentioned.

Lemma 3.1: *For any connected graph G on at least 3 nodes, if $\beta_c(G) = n$, then two adjacent nodes, each of degree 2, are always in a triangle of G.*

Proof: Suppose *x* and *y* are two adjacent nodes of degree two in *G*. Suppose to contrary that *x* and *y* are not in any triangle of *G*. Then, as *G* has at least 3 nodes, there exists a node *u* adjacent to *x* which is not adjacent to *y*. Let S' = *V* – {*x*, *y*}. Then, $d(x,u) \neq d(y,u)$ and $d(x,u) < d(x,s)$, for all *s* is S', so any cluster basis for *G* should have at most *n* – 2 nodes, which is a contradiction. ♦

Lemma 3.2: *For any connected graph G on at least 3 nodes, if* $\beta_c(G) = n$, *then G has no pendent node.*

Proof: Suppose that *G* has a pendent edge *e* = *xy*. Without loss of generality we assume *x* be a pendent node. Since $n \geq 3$, and *G* is not a path, *x* can be rooted uniquely from *y* (in fact, *d*(*x*, *y*) < *d*(*x*, *s*), for every $s \in V - \{x\}$). Thus the node *x* can be excluded to form a cluster basis *S* for the graph *G* (note that if *y* is adjacent to any other pendent node, then that node may be included, and no cluster basis includes only the pendent nodes by leaving its support node) and hence $\beta_c(G) \leq n-1$, a contradiction. ♦

Lemma 3.3: *For any connected graph G on at least 3 nodes, if* $\beta_c(G) = n-1$, *then every node of degree at least 2 should be included in the cluster basis of G.*

Proof: Follows immediately by noting the fact that the node of degree at least two is at a minimum distance 1 from at least two nodes. ♦

By the above lemma we immediately get the following theorem:

Theorem 3.4: *For any nontrivial connected graph G having at least 3 nodes, if every edge of G lies in a triangle of G and G has no induced subgraph isomorphic to the graph H of figure 3.1 below, then* $\beta_c(G) = n$.

H:

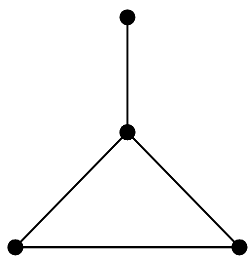

Figure 3.1

Proof: Suppose to contrary that $\beta_c(G) < n$. Let *S* be a cluster basis for *G*. Then there exists s_1 in the cluster basis such that $C[s_1]$ contains at least two nodes. Let $x, y \in C[s_1]$.

Case (i): *x* and *y* are adjacent.

Since every edge of *G* lies in a triangle of *G*, there exists *z* such that *xz* and *yz* are edges in *G*.

Claim: $z \in C[s_1]$

If not, then there exists $s_2 \in S$ such that $z \in C[s_2]$. But then

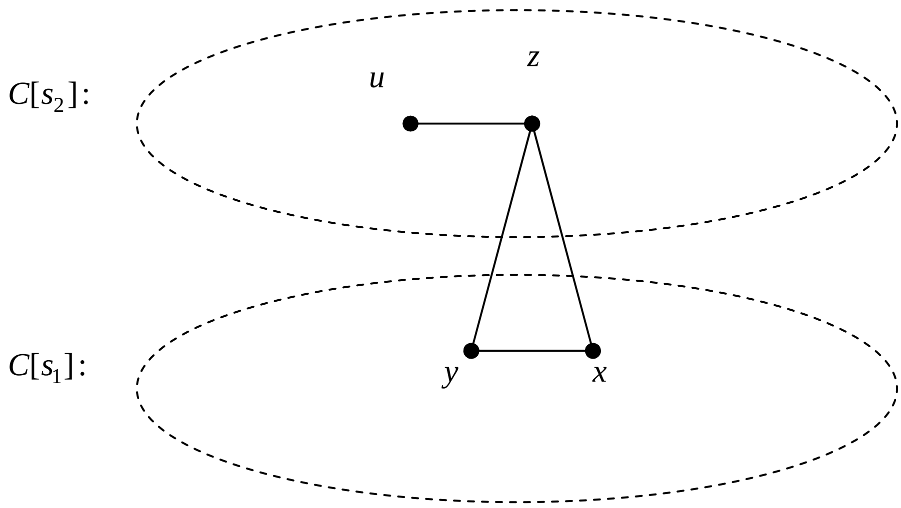

Figure 3.2(a)

Let $d(s_1, z) = a$ and $d(s_2, z) = b$. Then $b < a$ (since $z \in C[s_2]$) and $d(s_1, x) = a$ or $a - 1$ and $d(s_2, x) = b$ or $b + 1$ (here $a + 1$ or $b - 1$ are not admissible by Lemma 2.3). If $d(s_1, x) = a$ and $d(s_2, x) = b$, then $a < b$, which is not admissible (since already $b < a$). Or else if $d(s_1, x) = a$ and $d(s_2, x) = b + 1$, then $b < a < b + 1$ implies that $a \notin Z^+ \cup \{0\}$. Similarly, if $d(s_1, x) = a - 1$ and $d(s_2, x) = b$, then $a - 1 < b \Rightarrow a < b + 1 \Rightarrow b < a < b + 1$, which is again impossible. Thus, $d(s_1, x) = a - 1$ and $d(s_2, x) = b + 1$, but then $a - 1 < b + 1 \Rightarrow b < a < b + 2 \Rightarrow a = b + 1$. Thus there is one and only one possibility for *x*, *y* and *z* is that $d(s_1, z) = a$, $d(s_1, x) = d(s_1, y) = a - 1$, $d(s_2, z) = a - 1$, $d(s_2, x) = d(s_2, y) = a$. But then $a \neq 0$ (since each cluster contains a unique router), and hence there exists a node u in the shortest path from s_2 to z adjacent to z. Since u is in a shortest path, we have $d(s_2, u) = a - 2$, so u can not be adjacent to x or y. Hence the subgraph induced by the nodes $\{x, y, z, u\}$ is isomorphic to the graph H of figure 3.2(a), which is a contradiction. Hence the claim. Thus, all the nodes of the triangle are in the same cluster.

Case (ii): x and y are non adjacent.

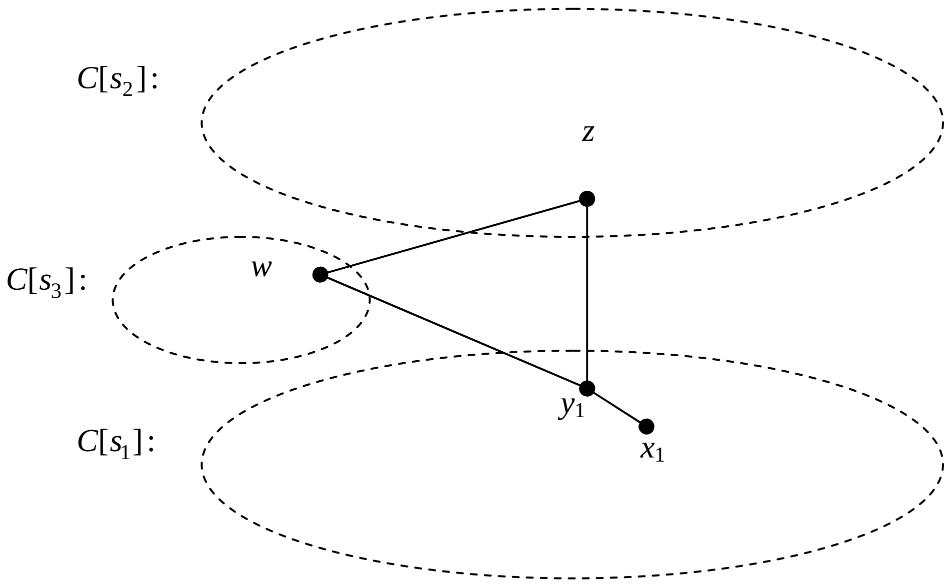

Figure 3.2(b)

Since G is connected and contains at least two clusters, there exists a node $y_1 \in C[s_1]$ and an adjacent node $z \in C[s_2]$, for some $s_2 \in S$. Now, as each cluster is a connected subgraph (since every node of it can be routed by the router present in it) of G, there is a path from x to y_1 in $C[s_1]$, so there is a node x_1 adjacent to y_1 in $C[s_1]$. We note here, by the case (i), that the edge $zx_1 \notin E$. The edge zy_1 must be in some triangle of G, so there exists w not in

$C[s_2] \cup C[s_1]$ such that w is adjacent to both z and y_1. But then, w cannot be adjacent to x_1 (as the edge x_1y_1 is in $C[s_1]$ and by the above case. Thus the set $\{w, z, x_1, y_1\}$ of nodes induce a subgraph isomorphic to H of figure 3.2(b), which is a contradiction. ♦

The converse of the above theorem 3.4 need not be true. The following theorem gives a counter example for one such graph namely the wheel.

Theorem 3.5: *For a wheel Wn , on n ≥ 3 nodes, the cluster dimension is n.*

Proof: Since the central node of a wheel is a full degree node, it should be in the cluster basis *S*. As the cluster dimension is at least two, one of the rim nodes to be in *S*. Let *v* be a rim node in *S* and *u* be a rim node adjacent to it. Then {*v*, *c*, *u*} induce a triangle with two nodes in two different clusters, thus by the case (i) in the proof of the above Theorem 3.4, *u* should be in *S*. Continuing this argument for the rim node *u*, and then its adjacent vertex, we can conclude that *S* = *V*. ♦

Theorem 3.6: *For a complete bipartite graph* $G = K_{m,n}$ *, with m ≥ n*

$$\beta_c(K_{m.n}) = \begin{cases} 1, & \text{if } m = n = 1 \\ 2, & \text{if } m = n = 2 \\ m+n-1, & \text{if } m \neq 1 \text{ and } n = 1 \\ m+n, & \text{otherwise} \end{cases}.$$

Proof: If *m* = *n* = 1, then the graph *G* is a path and hence $\beta_c(G) = 1$. If *m* = *n* = 2, then *G* = C_4, so the set $S = \{a, b\}$, where *a* and *b* are any two adjacent nodes of C_4 constitute a cluster basis.

If the graph is a star then with respect to the set $S = V - \{v\}$, where *v* is any pendent node, a node *x* ∈ *S* together with any other node *v* satisfies the properties $|d(x,x) - d(x,v)| > 0$ and $d(c,v) < d(z,v)$ for all *z* ∈ *S*, where *c* is the central node of the star. Hence it follows that $\beta_c(G) \leq n-1$. Further, if $\beta_c(G) < n-1$, then there exists two nodes *u* and *v* in *G* that are not in the cluster basis *S*. But, if both *u* and *v* are pendent nodes, then $d(x,u) = d(x,v)$ = 2 (or 1) from any node (central node) *x* in *S*. Else if one of the nodes *u* or *v* is a central node, then that node is at a minimum distance from more than two nodes in *S*. Hence any subset with fewer than n – 1 nodes constitute a cluster basis. Thus, $\beta_c(G) = n - 1$.

Finally, if m, n > 2 and $m + n > 4$, then G is a complete bipartite graph with at least five nodes, which is not a star or a cycle on 4 nodes. In this case if a node v is not in any subset S of V is adjacent to at least two nodes in S, so S can not constitute a cluster basis. Hence $S = V$.

Theorem 3.7: *If a graph G contains a bridge e = xy and every node of each component of G – e is need for a cluster basis. Then* $\beta_c(G) = n$.

Proof: As the nodes x and y are in different components of G – e, these two nodes are necessary for the cluster basis of the corresponding components. Hence x is adjacent to two basis element or equidistant from every pair of elements in a cluster basis of the component it lies. Similar argument holds for the node y also. Now, every path in G from a node y to a node in a component containing x should contain the bridge xy it follows that the node y is also at a equidistant from every pair of node in that components. Hence y cannot be excluded to form any cluster basis. Similarly x can not be excluded. Thus cluster basis should contain all the nodes.

4. GRAPHS OF CLUSTER DIMENSIONS *K* AND DIAMETER *D.*

In this section we give an upper bound for number of nodes in the network with prescribed number of routers in terms of its diameter.

Theorem 4.1: *If the cluster dimension of a graph of diameter d is k, then*

$$|V| \le k + d^k - \sum_{j=1}^{d}\sum_{l=2}^{k}\binom{k}{k-l}(d-j)^{k-l} .$$

Proof: Let $S = \{s_1, s_2, \ldots, s_k\}$ be a cluster basis for G. For each node v of the graph, we can associate a vector $(v_1, v_2, \ldots, v_k)$, where v_i , for each i, $1 \le i \le k$, denotes the distance from the route node s_i to the vertex v. Then, as the diameter is d, each v_i takes the values form the set $T = \{1, 2, \ldots, d\}$. Since S is a cluster basis, the vectors associated to two distinct vertices should be distinct. The maximum number of distinct vectors can be created to assign nodes form the set T is d^k . Further, by the definition of cluster basis, it follows that the vector associated to each node has exactly one component which is strictly smaller than all other components of it. Hence if $v_i = 1$ for any node, then $v_j \geq 2$. If 1 is repeated in l times in a vector, where $2 \leq l \leq k$, then the number of vectors having remaining components $k - l$ components are all greater than 1 is $\binom{k}{k-l}(d-1)^{k-l}$. Thus number of vectors of dimension k with repeated 1 is $\sum_{l=2}^{k}\binom{k}{k-l}(d-1)^{k-l}$. In general, The number of vectors of dimension k with repeated j, $1 \leq j \leq d$, and all other components more than j is $\sum_{l=2}^{k}\binom{k}{k-l}(d-j)^{k-l}$. Thus the total number such vectors is $\sum_{j=1}^{d}\sum_{l=2}^{k}\binom{k}{k-l}(d-j)^{k-l}$.

Further, the vector associated to the routers only contains 0 in exactly one of the components. Thus, possible number of valid vectors that can be associated to a node of a graph is $k + d^k - \sum_{j=1}^{d}\sum_{l=2}^{k}\binom{k}{k-l}(d-j)^{k-l}$.

Remark 4.2: The upper bound shown in the above Theorem 4.1 is tight. In fact, for the complete graph on n nodes, $k = n$ and $d = 1$, so $k + d^k - \sum_{j=1}^{d}\sum_{l=2}^{k}\binom{k}{k-l}(d-j)^{k-l} =$

$$n+1-\sum_{j=1}^{1}\sum_{l=2}^{n}\binom{n}{n-l}(1-j)^{n-l} \;=\; n+1-\sum_{j=1}^{1}\left\{\binom{n}{n-2}(1-j)^{n-2}+\cdots+\binom{n}{1}(1-j)^{1}+\binom{n}{0}(1-j)^{0}\right\} \;=$$

$$n+1-\sum_{j=1}^{1}\{1\} = n = |V|.$$

5. ALGORITHM TO DETERMINE CLUSTER DIMENSION AND CLUSTER BASIS

We now give an efficient algorithm to determine the cluster dimension of a given network. The input of the network is taken in terms of distance between two nodes rather than the adjacency or incidence. The purpose of representation of a network in terms of its distance matrix can be easily adopted for any type of network because the distance can be considered in various ways depending on the purpose in hand. For example the distance between two stations can be considered as air distance or for the land lines it may be length of the wire connection between two junctions to identify the fault in an exact location. The input can be easily modified for a weighted graph by considering the sum of weights on the edges in a shortest path as the distance between the corresponding nodes in the network.

Throughout the algorithm we assume the network is a connected and represented by a square matrix $D = (d_{ij})$ of order n, where n is the number of nodes in the network and d_{ij} denotes the distance between the two nodes i and j. For any network the matrix D is a symmetric matrix with zero diagonal entry if and only if the network is undirected.

Step 0: **Best case analysis**

If G is a path on n nodes (that is $\sum_{i=1}^{n} d_{ij} = \binom{n}{2}$) for some j) , then return Cdim = 1 and Cbas = {end node}

Step 1: **Set Selection : initialization**

For $k = 1$ to $n - 1$, (n is the number of nodes)

Consider the k – element subsets $D_1, D_2, \ldots, D_{\binom{n}{k}}$, of nodes.

Step 2: **Labeling: initialization**

For $i = 1$ to $\binom{n}{k}$

Step 3: **Re-label** the nodes in D_i, by the set $\{v_1, v_2, \ldots, v_k\}$ and others by the set $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$.

Step 4**: ID Generation**

If $k < (n-1)$

For $r = 1$ to k

For $p = k + 1$ to $n - 1$

For $q = p + 1$ to n

Store $[r]\,[p]\,[q] = \left| d_{rp} - d_{rq} \right|$

Step 5: **ID Verification**

For $l = k + 1$ to $n - 1$

For $m = l + 1$ to n

Check ← 0

For $r = 1$ to k

Check ← Check + store[r][l][m]

If check = 0, increment i and go to step 5

Go to step 7

Step 6: **Verification of exhaustiveness**

Test whether new D_i exists, if exists, then go to step 3

Else if $k < n - 1$ return to step 1 to increment k.

Step 7: **Verification of uniqueness**

For $j = k + 1$ to n

$\text{Min}\left[j\right] = \underset{i}{\text{Min}}\{a_{ij}\}$

$\text{Next Min}\left[j\right] = \underset{i}{\text{Min}}\{\{a_{ij}\} - Min[j]\}$

If $(\text{Min}\left[j\right] - \text{Next Min}\left[j\right]) \neq 0$, then return k, the set D_i. Exit

If $k < n - 1$, return to step 6

Return n, the set V. Exit

Open problem: Characterize the graph on n nodes whose cluster dimension is n?

Conjecture: For any graph G on at least three nodes, $\beta_c(G) = n$ if and only if one of the following holds;

(i) Every edge of G is in a triangle and G cannot have an induced subgraph isomorphic to the graph H of figure 5.1 below.

(ii) G is a complete bipartite graph $K_{m,n}$, with m, n > 2 and m + n >4.

(iii) If G has an edge e which is not in any triangle of G, then $\beta_c(G - e) = n$.

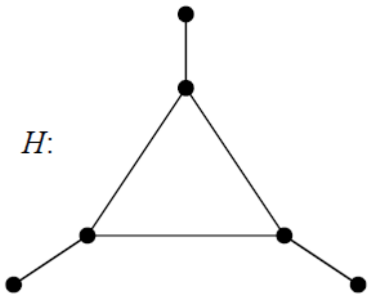

Figure 5.1

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