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A THEOREM ON MODULES WITH FINITE GOLDIE DIMENSION

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Abstract. The concepts: 'complement' and 'finite Goldie dimension' in the theory of modules (over rings) are well known. The finite Goldie dimension of a submodule N of a module is usually denoted by 'dim N'. The object of the present paper is to prove: if M is a module with FGD; K_1, K_2 are submodules of M; and $K_1 \cap K_2$ is a complement in M, then the condition: $\dim K_1 + \dim K_2 = \dim(K_1 + K_2) + \dim(K_1 \cap K_2)$ is true.

1. Introduction

It is well known that the dimension of a vector space is defined as the number of elements in the basis. One can define a basis of a vector space as a maximal set of linearly independent vectors or a minimal set of vectors, which span the space. The former, when generalized to modules over rings, becomes the concept of Goldie Dimension. The concept of Goldie Dimension in modules was studied by several authors like Anh, Marki, Camillo, Zelmanowitz, Goldie, Reddy and Satyanarayana (cf. [1 to 10]). Let R be a fixed (not necessarily commutative) ring. Throughout this paper, we are concerned with left R-modules M. Like in

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Goldie [6], we shall use the following terminology. A non-zero submodule K of M is called *essential* in M (or M is an *essential extension* of K) if $K \cap A = (0)$ for any other submodule A of M, implies $A = (0)$. M has *Finite Goldie Dimension* (abbr. FGD) if M does not contain a direct sum of infinite number of non-zero submodules. Equivalently, M has a finite Goldie dimension if for any strictly increasing sequence $H_0 \subset H_1 \subset H_2 \subset \cdots$ of submodules of M, there is an integer i such that H_k is essential submodule in H_{k+1} for every $k \geq i$. A non-zero submodule H of M is *uniform*, if every non-zero submodule of H is essential in H. Then it is proved (cf. [6]) that in any module M with FGD, there exist uniform submodules $U_1, U_2, \ldots, U_n$ whose sum is direct and essential in M. The number n is independent of the uniform submodules. This number n is called the *Goldie dimension* of M and is denoted by $\dim M$. It can be easily proved that if M has FGD, then every submodule K of M has also FGD and $\dim K \leq \dim M$. Furthermore, if K, A are sbumodules of M, and K is a maximal submodule of M such that $K \cap A = (0)$, then we say that K is a *complement* of A (or a complement in M). It is easy to prove that if K is a submodule of M, then K is a complement in $M \Leftrightarrow$ there exists a submodule A in M such that $A \cap K = (0)$ and $K^1 \cap A \neq (0)$ for any submodule K^1 of M such that K^1 properly contains K. In this case, we have $K + A$ is essential in M. It is proved that (cf. [7]) that if M has FGD, then a submodule K is a complement $\Leftrightarrow$ M/K has FGD and $\dim(M/K) = \dim M - \dim K$.

The purpose of this note is to prove the following results.

Main Theorem. *If M has FGD and K_1, K_2 are two submodules of M such that $K = K_1 \cap K_2$ is a complement, then* $\dim K_1 + \dim K_2 = \dim(K_1 + K_2) + \dim(K_1 \cap K_2)$.

This main theorem is an immediate consequence of a theorem at the bottom of Page 250 of [5]. But the proof we presented here is totally different from the very general and long proof of [4] and [5].

2. Some Results

For some fundamental results used in this paper, we refer (cf. [7]).

Now we state a Result and a Corollary. In what follows M will always mean a module.

Result 2.1. (See 17.3 on Page 138 of [10]) *Let K, M, L be R-modules. If $h : K \to M$ is a homomorphism and $L \leq_e M$, then $h^{-1}(L) \leq_e K$.*

Corollary 2.2. *Let K be a submodules of a module M.*
$\pi : M \to M/K$ be the canonical epimorphism. If $\pi(S) \leq_e \pi(M)$, then $S + K \leq_e M$.

Proof. Follows from above Result 2.1.

Note 2.3. The converse of the above Corollary is not true. That is, there exist a module M, two submodules S and K of M such that $S \leq_e M$ but $\pi(S)$ is not essential in M/K, where π is the canonical mapping from M to M/K.

Example 2.4. Write $M = Z$, the group of integers, $R = Z$ the ring of integers, $S = 2Z$, $K = 6Z$. Now M is a module over R. S and K are submodules of M. Consider the canonical epimorphism $\pi : M \to M/K$. Now $S \leq_e M$ and $\pi(S) = \pi(2Z) = 2Z/6Z = \{\overline{0}, \overline{2}, \overline{4}\}$ is not essential in $M/K = Z_6 = \{\overline{0}, \overline{3}\} \oplus \{\overline{0}, \overline{2}, \overline{4}\}$.

Theorem 2.5. *Suppose M have FGD and K_1, K_2 are two submodules of M such that $K = K_1 \cap K_2$ is a complement. Then* $\dim K_1 + \dim K_2 = \dim(K_1 + K_2) + \dim(K_1 \cap K_2)$.

Proof. Let A be a complement of K in K_1, and B be a complement of K in K_2. Then $A \oplus K \leq_e K_1$ and $B \oplus K \leq_e K_2$

$$\Rightarrow (A + K)/K \leq_e K_1/K \text{ and } (B + K)/K \leq_e K_2/K \text{ (by Theorem 1 of [7]).}$$

Now $(K_1/K) \cap (K_2/K) = \dfrac{K_1 \cap K_2}{K} = (0)$.
We have that

$$\begin{aligned}
&(A + B + K)/K = (A + K)/K \oplus (B + K)/K \leq_e \frac{K_1}{K} \oplus \frac{K_2}{K} = \frac{K_1 + K_2}{K} \\
&\Rightarrow (A + B + K)/K \leq_e (K_1 + K_2)/K \\
&\Rightarrow A + B + K \leq_e K_1 + K_2 \quad \text{(by Corollary 2.2)} \\
&\Rightarrow \dim(A + B + K) = \dim(K_1 + K_2).
\end{aligned}$$

Now we verify the sum $A + B + K$ is direct. Let $a + b + k = 0$ for some $a \in A$, $b \in B$, $k \in K$. It follows that $b = -a - k \in K_1 \cap K_2 = K$. Then $b \in B \cap K = 0$, hence $b = 0$. Now $a \in A \cap K = 0$, hence $a = 0$ and then $k = 0$. Thus the sum $A + B + K$ is direct. Since $A \oplus B \oplus K \leq_e K_1 + K_2$, we have that

$$\begin{aligned}\dim(K_1 + K_2) &= \dim(A \oplus B \oplus K)\\ &= \dim A + \dim B + \dim K\\ &= (\dim K_1 - \dim K) + (\dim K_2 - \dim K) + \dim K\\ &= \dim K_1 + \dim K_2 - \dim K\\ &= \dim K_1 + \dim K_2 - \dim(K_1 \cap K_2).\end{aligned}$$

This completes the proof.

As an application to the vector spaces, we have the following:

Corollary 2.6. *Suppose V is a finite dimensional vector space. Then every subspace W of V is a complement submodule of V when we consider V as a module over the same field. From Theorem 2.5, we can conclude that for any two subspaces K_1 and K_2 of V, we have that* $\dim K_1 + \dim K_2 = \dim(K_1 + K_2) + \dim(K_1 \cap K_2)$.

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