On Fuzzy Cosets of Gamma Nearrings

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Abstract

In this paper, we consider fuzzy notion of a Γ-near ring, introduce the notion of a fuzzy coset and obtained some related important fundamental isomorphism theorems.

Key Words: Gamma nearring, ideal, fuzzy ideal, fuzzy coset, Gamma nearring homomorphism.

Introduction

A non-empty set $N$ with two binary operations $+$ and $\cdot$ is called a nearring if it satisfies the following axioms.

(i) $(N, +)$ is a group (not necessarily Abelian);
(ii) $(N, \cdot)$ is a semi-group;
(iii) $(a + b)c = ac + bc$ for all $a, b, c \in N$.

Precisely speaking, it is a right near ring. Moreover, a near ring $N$ is said to be a zero-symmetric nearring if $n0 = 0$ for all $n \in N$, where 0 is the additive identity in $N$.

The concept of Γ-nearring, a generalization of both the concepts nearring and Γ-ring was introduced by Satyanarayana [10]. Later, several authors such as Satyanarayana [11, 12], Booth [1-3] and Booth and Groenewald [4] studied the ideal theory of Γ-nearrings.

Let $(M, +)$ be a group (not necessarily Abelian) and $\Gamma$ a non-empty set. Then $M$ is said to be a Γ-nearring if there exists a mapping $M \times \Gamma \times M \to M$ (the image of $(a, \alpha, b)$ is denoted by $a\alpha b$), satisfying the following conditions:

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(i) \((a + b)ac = aac + bac\);
(ii) \((aob)\beta c = a\alpha(b\beta c)\) for all \(a, b, c \in M\) and \(\alpha, \beta \in \Gamma\). Moreover, \(M\) is said to be zero-symmetric if \(aa0 = 0\) for all \(a \in M\) and \(\alpha \in \Gamma\), where 0 is the additive identity in \(M\). A normal subgroup \((I, +)\) of \((M, +)\) is called
(i) a left ideal, if \(a(a(b + i) = a(b + i)\) for all \(a, b \in M\), \(i \in I\);
(ii) a right ideal, if \(i(a = a(i)\) for all \(a \in M\), \(i \in I\);
(iii) an ideal, if it is both a left and a right ideal.

It is clear that if \(M\) is a \(\Gamma\)-nearring, then the elements of \(\Gamma\) act as binary operations on \(M\) such that the system \((M, +, \gamma)\) is a nearring for all \(\gamma \in \Gamma\). The relations between the concepts \(\Gamma\)-nearring and nearring were studied in Section 1 of Satyanarayana [12]. Throughout this paper, \(M\) stands for a zero-symmetric \(\Gamma\)-nearring. The ideal generated by an element \(a \in M\) is denoted by \(<a>\). For other definitions and preliminary results on \(\Gamma\)-nearrings we refer to [7, 11, 12].

The concept of fuzzy subset was introduced by Zadeh [14]. A fuzzy set in a set \(A\) is a function \(\mu: A \rightarrow [0, 1]\). For any \(t \in [0, 1]\), the set \(\mu_t\) defined by \(\mu_t = \{x \in A | \mu(x) \geq t\}\) is called as a level subset of \(\mu\). For any two fuzzy sets \(\mu, \sigma\) in \(A\), we write \(\mu \subseteq \sigma\) if \(\mu(x) \leq \sigma(x)\) for all \(x \in A\). (In this case, we also say that \(\mu\) is a subset of \(\sigma\).) Let \(X\) and \(Y\) be two non-empty sets, \(f: X \rightarrow Y\), \(\mu\) and \(\sigma\) be fuzzy subsets of \(X\) and \(Y\) respectively. Then \(f(\mu)\), the image of \(\mu\) under \(f\) is a fuzzy subset of \(Y\) defined by

\[
(f(\mu))(y) = \begin{cases} 
\sup_{f(x) = y} \mu(x) & \text{if } f^{-1}(y) \neq \phi \\
0 & \text{if } f^{-1}(y) = \phi.
\end{cases}
\]

\(f^{-1}(\sigma)\), the preimage of \(\sigma\) under \(f\) is a fuzzy subset of \(X\) defined by \((f^{-1}(\sigma))(x) = \sigma(f(x))\) for all \(x \in X\).

Jun. Sapanci and Ozturk [7] introduced the concept “fuzzy ideal” in \(\Gamma\)-nearrings and studied some fundamental properties. It is clear that a fuzzy set \(\mu\) in a \(\Gamma\)-nearring \(M\) is a mapping \(f: M \rightarrow [0, 1]\).

A fuzzy set \(\mu\) in a \(\Gamma\)-nearring \(M\) is called a fuzzy left (resp., right) ideal of (or in) \(M\) if
(i) \(\mu\) is a fuzzy normal subgroup with respect to addition (that is, \(\mu(x - y) \geq \min\{\mu(x), \mu(y)\}\), and \(\mu(y + x - y) \geq \mu(x)\));
(ii) \(\mu(x\alpha(y+z)-x\alpha y) \geq \mu(z)\) (resp., \(\mu(x\alpha y) \geq \mu(x)\)) for all \(x, y, z \in M\) and \(\alpha \in \Gamma\).
If \( \mu \) is both left and right ideal, then \( \mu \) is said to be a fuzzy ideal in \( M \).

It is easy to verify that if \( \mu \) is a fuzzy ideal of \( M \), then the following three conditions hold:

(i) \( \mu(0) \geq \mu(x) \) (in other words, \( \mu(0) = \max \{ \mu(x) \mid x \in M \} \));

(ii) \( \mu(x + y) = \mu(y + x) \); and

(iii) \( \mu(x - y) = \mu(0) \) implies \( \mu(x) = \mu(y) \), for all \( x, y \in M \).

Jun, Sapanci and Ozturk [7] proved that for a fuzzy set \( \mu \) in \( M \), \( \mu \) is a fuzzy left (resp., right) ideal of \( M \) if and only if each level subset \( \mu_t, t \in \text{im} \ (\mu) \), of \( \mu \) is a left (resp. right) ideal of \( M \).

A fuzzy (left, right) ideal \( \mu \) of \( M \) with \( \mu(0) = 1 \) is called a normal (left, right) ideal of \( M \). Jun, Kim and Ozturk [5] introduced the fuzzy maximal ideals and some related properties were studied. A fuzzy ideal \( \mu \) of \( M \) is said to be a fuzzy maximal ideal if it satisfies the two conditions: (i) \( \mu \) is non-constant; and (ii) \( \mu^* \) is a maximal element among all the normal fuzzy ideals of \( M \) where \( \mu^* (x) = \mu(x) + 1 - \mu(0) \), for all \( x \in M \).

For other preliminary definitions and results related to fuzziness, see [7].

This paper is divided into three sections. In Section 1, we prove a result on fuzzy ideals.

In Section 2, we introduce the concept fuzzy coset in \( \Gamma \)-nearrings and prove that the set of all fuzzy cosets forms a \( \Gamma \)-nearring (theorem 2.4). In the last section, that is, in Section 3, we prove the following important fundamental results related to isomorphism theorems on \( \Gamma \)-nearrings:

(i). \( M/\mu \) is isomorphic to the quotient \( \Gamma \)-nearring \( M/M_{\mu} \), where \( M/\mu \) is the \( \Gamma \)-nearring of all the cosets of \( M \) with respect to the fuzzy ideal \( \mu \), and

\[
M_{\mu} = \{ x \in M/\mu(x) = \mu(0) \}.
\]

(ii) There exists an order preserving bijection between the set \( P \) of all fuzzy ideals \( \sigma \) of \( M \) such that \( \sigma \supseteq \mu \) and \( \sigma(0) = \mu(0) \) and the set \( Q \) of all fuzzy ideals \( \theta \) of \( M/\mu \) such that \( \theta \supseteq \theta_{\mu} \), where \( \theta_{\mu} \) is a fuzzy ideal of \( M/\mu \) defined by \( \theta_{\mu}(x+\mu) = \mu(x) \) for all \( x \in M \).

(iii) Let \( h: M \to M^1 \) be a \( \Gamma \)-nearring epimorphism and let \( \sigma \) be a fuzzy ideal of \( M^1 \) and \( \mu = h^{-1}(\sigma) \). Then the map \( \psi: M/\mu \to M^1/\sigma \) defined by \( \psi(x+\mu) = h(x) + \sigma \) is a \( \Gamma \)-near ring isomorphism.

As a consequence of (iii), we obtain the following result: If \( \mu \) and \( \sigma \) are two fuzzy ideals of \( M \) such that \( \mu \subseteq \sigma \) and \( \sigma(0) = \mu(0) \), then \( M/\sigma \cong (M/\mu)/(\sigma/\mu) \).
1. Fuzzy ideals

Theorem 1.1 If $\mu$ is a fuzzy ideal of $M$, and $a \in M$ then $\mu(x) \geq \mu(a)$ for all $x \in <a>$. 

Proof. By straightforward verification, we conclude that for $a \in M$, $<a> = \bigcup_{i=0}^{\infty} A_i$, where $A_{k+1} = A_k^+ \cup A_k^0 \cup A_k^{++}$, $A_0 = \{a\}$ and

- $A_k^+ = \{n+x-n / n \in N, x \in A_k\}$;
- $A_k^0 = \{n_1\alpha(n_2+a) - n_1\alpha n_2 / n_1, n_2 \in M, a \in A_k, \alpha \in M\}$;
- $A_k^{++} = \{x-y / x, y \in A_k\}$;
- $A_k^0 = \{x \in A_k, \alpha \in \Gamma \text{ and } m \in M\}$.

We prove that $\mu(u) \geq \mu(a)$ for all $u \in A_m$ for $m \geq 1$. For this, we use induction on $m$. It is obvious if $m = 0$. Suppose the induction hypothesis for $k$. That is, $\mu(x) \geq \mu(a)$ for all $x \in A_k$. Now let $v \in A_k^+ \cup A_k^0 \cup A_k^{++}$. Suppose $v \in A_k^+$. Then $v = n+x-n$ for some $x \in A_k$. Now $\mu(v) = \mu(n+x-n) \geq \mu(x)$ (since $\mu$ is a fuzzy ideal of $N$) $\geq \mu(a)$ (by induction hypothesis). Let $v \in A_k^0$. Then $v = x_1-x_2$ for some $x_1, x_2 \in A_k$. Now $\mu(v) = \mu(x_1-x_2) \geq \min (\mu(x_1), \mu(x_2)) \geq \mu(a)$, by induction hypothesis.

Suppose that $v \in A_k^{++}$. Then $v = n_1\alpha(n_2+a) - n_1\alpha n_2$ for some $n_1, n_2 \in M, x \in A_k$ and $\alpha \in \Gamma$. Now $\mu(v) = \mu(n_1\alpha(n_2+x) - n_1\alpha x) \geq \mu(x)$ (since $\mu$ is a fuzzy ideal of $M$) $\geq \mu(a)$ (by induction hypothesis on $k$).

Suppose $v \in A_k^0$. Then $v = x \alpha m$ for some $x \in A_k, \alpha \in \Gamma$ and $m \in M$. Now $\mu(v) = \mu(x \alpha m) \geq \mu(x)$ (since $\mu$ is a fuzzy ideal of $M$) $\geq \mu(a)$ (by induction hypothesis on $k$).

Thus in all cases we proved that $\mu(v) \geq \mu(a)$ for all $v \in A_{k+1}$. Hence by the principle of mathematical induction, we conclude that $\mu(v) \geq \mu(a)$ for all $v \in A_m$ and for all positive integers $m$. Hence $\mu(x) \geq \mu(a)$ for all $x \in <a>$. \hfill \( \Box \)

Corollary 1.2 If $I$ is an ideal of $N$ with $I = <a> = <b>$, then $\mu(a) = \mu(b)$.

Proof. Since $a \in <b>$ and $b \in <a>$, we have $\mu(a) \geq \mu(b)$ and $\mu(b) \geq \mu(a)$, so $\mu(a) = \mu(b)$. \hfill \( \Box \)
2. Fuzzy Cosets

**Definition 2.1** Let $\mu$ be a fuzzy ideal of $M$ and $m \in M$. Then the fuzzy subset $m+\mu$ defined by $(m+\mu)(m^1) = \mu(m^1+m)$ for all $m^1 \in M$ is called a fuzzy coset of the fuzzy ideal $\mu$.

**Lemma 2.2** Let $\mu$ be a fuzzy ideal of $M$. Then for $x, y, z \in M$ we have the following:

(i) $x+\mu = y+\mu$ if and only if $\mu(x-y) = \mu(0)$;
(ii) If $x + \mu = y + \mu$, then $\mu(x) = \mu(y)$;
(iii) $\mu(x+y) = \mu(y+x)$;
(iv) $M_\mu = \{x \in M / \mu(x) = \mu(0)\}$ is an ideal of $M$;
(v) Every fuzzy coset of a fuzzy ideal $\mu$ of $M$ is constant on $M_\mu$;
(vi) If $z \in M_\mu$, then $(x+\mu)(z) = \mu(x)$.

**Proof.** (i), (ii), (iii) have straightforward verifications.

(iv) Proved in Jun, Sapanci and Ozturk [7].

(v) Let $y, z \in M_\mu$. We show that $(x+\mu)(y) = (x+\mu)(z)$. Since $y, z \in M_\mu$, we have that $\mu(y) = \mu(0)$ and $\mu(z) = \mu(0)$. Since $M_\mu$ is an ideal, we have that $y-z \in M_\mu$. So $\mu(y-z) = \mu(0)$. Now $(x+\mu)(y) = \mu(y-x)$ (by the definition of fuzzy coset)

\[
\begin{align*}
&= \mu(-(y-x)) \text{ (since } \mu \text{ is a fuzzy ideal of M)} \\
&= \mu(y-x) = \mu(-x+y+x+z) \text{ (since } \mu \text{ is a fuzzy normal subgroup)} \\
&\geq \min \{\mu(-x+y), \mu(x-z)\} \text{ (since } \mu \text{ is a fuzzy ideal of M)} \\
&= \min \{\mu(y-x), \mu(x-z)\} = \min \{\mu(0), \mu(x-z)\} \text{ (since } \mu(y-x) = \mu(0)) \\
&= \mu(x-z) \text{ (since } \mu(0) \geq \mu(x-z))
\end{align*}
\]

Therefore $(x+\mu)(y) \geq (x+\mu)(z)$.

Similarly by interchanging $y$ and $z$ in above part, we can show that $(x+\mu)(z) \geq (x+\mu)(y)$. Hence $(x+\mu)(y) = (x+\mu)(z)$ for all $y, z \in M_\mu$.

(vi) Let $z \in M_\mu$. Then $\mu(z) = \mu(0)$. Since $z, 0 \in M_\mu$, we have $(x+\mu)(z) = (x+\mu)(0)$ (by (vi)) $\Rightarrow \mu(z-x) = \mu(0-x) = \mu(-x) = \mu(x)$ (since $\mu$ is a fuzzy ideal of $M$). Therefore $\mu(z-x) = \mu(x)$. Hence $(x+\mu)(z) = \mu(x)$. \[\square\]

**Notation 2.3** We write $M/\mu = \{m + \mu \mid m \in M\}$, the set of all fuzzy cosets of $\mu$. 

Theorem 2.4 Let $\mu$ be a fuzzy ideal of $M$. Then the set $M/\mu$ of all fuzzy cosets of $\mu$ is a $\Gamma$-nearring with respect to the operations defined by 

$$(x+\mu)+(y+\mu) = (x+y)+\mu; \text{ and } (x+\mu)\alpha(y+\mu) = xy+\mu \text{ for all } x, y \in M \text{ and } \alpha \in \Gamma.$$ 

Proof. First we verify that “+” is well defined. Suppose $x+\mu = u+\mu$, $y+\mu = v+\mu$. Then by Lemma 2.2 (i), $\mu(x-u) = \mu(y-v) = \mu(0)$. Now $\mu\{(x+y)-(u+v)\} = \mu\{(x+y-v-u)\} = \mu\{-u+(x+y-v)\} = \mu\{(-u+x)+(y-v)\} \geq \min \{\mu(-u+x), \mu(y-v)\}$ (since $\mu$ is a fuzzy ideal of $M$) = $\mu(0)$. Also it is clear that $\mu(0) \geq \mu\{(x+y)-(u+v)\}$. Therefore $\mu\{(x+y)-(u+v)\} = \mu(0)$. Hence by Lemma 2.2 (i), $(x+y)+\mu = (u+v)+\mu$. This shows that “+” is well defined.

Next we verify that “.” is well defined. Now $\mu(xo\alpha-u\alpha v) = \mu(u\alpha v-xo\alpha) = \mu(u\alpha v+xo\alpha-xo\alpha) = \mu(u-x)\alpha+xo(y+(-y+v))-xo\alpha \geq \min \{\mu(u-x), \mu(-y+v)\}$ (since $\mu$ is a fuzzy ideal of $M$) = $\mu(0, 0) = \mu(0) \geq \mu(xo\alpha-u\alpha v)$. This shows that $\mu(xo\alpha-u\alpha v) = \mu(0)$. By Lemma 2.2 (i), $xo\alpha+\mu = u\alpha v+\mu$.

Now we verify that $M/\mu = \{x+\mu \mid x \in M\}$ is a $\Gamma$-nearring with respect to the above operations defined. A direct verification shows that $(M/\mu, +)$ is a group.

Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Now $((x+\mu)+(y+\mu))\alpha(z+\mu)$

$= ((x+y)+\mu)\alpha(z+\mu)$

$= ((x+y+\alpha z)+\mu) \text{ (by definition of multiplication)}$

$= ((xo\alpha)+(\alpha z)+\mu) \text{ (by right distributive law in } M)$

$= ((xo\alpha)+\mu+(\alpha z)+\mu)$

$= (x+\mu)\alpha(\alpha z)+\mu+(y+\mu)\alpha(z+\mu)$.

Also $((x+\mu)\alpha(y+\mu))\beta(z+\mu) = ((xo\alpha)+\mu)\beta(z+\mu) \text{ (by definition of addition)}$

$= (xo\alpha)\beta(z+\mu) = xo(y\beta z)+\mu = (x+\mu)\alpha(y\beta z)+\mu = (x+\mu)\alpha((y+\mu)\beta(z+\mu))$.

Hence $M/\mu$ is a $\Gamma$-nearring. $\Box$

Notation 2.5 Let $M$ be a fuzzy ideal. We define $\theta_\mu: M/\mu \to [0, 1]$ by $\theta_\mu(x + \mu) = \mu(x)$ for all $x \in M$.

Lemma 2.6 If $\mu$ is a fuzzy ideal, then $\theta_\mu$ is a fuzzy ideal of $M/\mu$.

Proof. Given that $\theta_\mu(x+\mu) = \mu(x)$. Suppose $x+\mu = y+\mu$. Then $\mu(x-y) = \mu(0)$. This implies $\mu(x) = \mu(y)$. That is, $\theta_\mu(x+\mu) = \theta_\mu(y+\mu)$. Therefore $\theta_\mu$ is well defined.

We verify that $\theta_\mu$ is a fuzzy ideal of $M/\mu$. 16
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(i) $\theta_\mu((x+\mu)+(y+\mu)) = \theta_\mu(x+y+\mu) = \mu(x+y)$ (by definition of $\theta_\mu$)
\geq \min \{\mu(x), \mu(y)\}$ (since $\mu$ is a fuzzy ideal of $M$

Therefore $\theta_\mu((x+\mu)+(y+\mu)) \geq \min \{\theta_\mu(x+\mu), \theta_\mu(y+\mu)\}$.

(ii) $\theta_\mu(x+\mu) = \mu(x) = \mu(-x)$ (since $\mu$ is a fuzzy ideal of $M$) $= \theta_\mu(-x+\mu)$, by definition

(iii) $\theta_\mu((x+\mu)+(y+\mu)-(y+\mu)) = \theta_\mu((x+y-x)+\mu) = \mu(x+y-x) = \mu(x) = \theta_\mu(x+\mu)$

(iv) $\theta_\mu((x+\mu)\alpha(y+\mu)) = \theta_\mu(x\alpha y+\mu) = \mu(x\alpha y) \geq \mu(x) = \theta_\mu(x+\mu)$, by definition

(v) $\theta_\mu\{((x+\mu)\alpha((y+\mu)+(z+\mu))-(x+\mu)\alpha(y+\mu))\} = \theta_\mu\{(x+\mu)\alpha((y+z)+\mu)-(x+\mu)\alpha(y+\mu)\}$
\[= \theta_\mu\{(x\alpha(y+z)+\mu)-(x\alpha y+\mu)\} = \theta_\mu\{(x\alpha(y+z)-(x\alpha y)+\mu)\}
\]
\[= \mu\{x\alpha(y+z)-(x\alpha y)\} \geq \mu(z) = \theta_\mu(z+\mu).
\]
Hence $\theta_\mu$ is a fuzzy ideal of $M/\mu$. \hfill \Box

3. Some Isomorphism Theorems

Theorem 3.1 (Jun, Sapanci and Ozturk [7]): If $\mu$ is a fuzzy (left, right) ideal of $M$ then the set $M_\mu = \{x \in M / \mu(x) = \mu(0)\}$ is a fuzzy (left, right) ideal of $M$.

Definition 3.2 Let $M$ and $N$ be $\Gamma$-nearrings. A map $\theta : M \to N$ is called a $\Gamma$-nearring homomorphism if $\theta(x + y) = \theta(x) + \theta(y)$ and $\theta(x\alpha y) = \theta(x)\alpha\theta(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover if $\theta$ is one-one (onto, bijection, respectively) then $\theta$ is called as monomorphism (epimorphism, isomorphism, respectively).

Now we prove the following theorem.

Theorem 3.3 If $\mu$ is a fuzzy ideal of $M$ then the map $\theta : M \to M/\mu$, defined by $\theta(x) = x+\mu$, $x \in M$, is a $\Gamma$-near-ring epimorphism with kernel $M_\mu$ where

$M_\mu = \{x \in M / \mu(x) = \mu(0)\}$. Moreover $M/M_\mu$ is isomorphic to $M/\mu$ (under the mapping $x+M_\mu \to x+\mu$).

Proof. $\theta(x+y) = \theta(x) + \theta(y)$ is clear.

Now $\theta(x\alpha y) = (x\alpha y)+\mu$ (by definition of $\theta$) $= (x+\mu)\alpha(y+\mu)$
\[= \theta(x)\alpha\theta(y). \text{ Therefore } \theta \text{ is a homomorphism.} \]
Now $x \in \text{kernel } \theta \Leftrightarrow \theta(x) = 0 = 0 + \mu \Leftrightarrow x + \mu = 0 + \mu \Leftrightarrow \mu(x - 0) = \mu(0)$, (by Lemma 2.2(i)) $\Leftrightarrow \mu(x) = \mu(0) \Leftrightarrow x \in M_{\mu}$. This shows that kernel $\theta = M_{\mu}$. \hfill $\square$

**Notation 3.4** Let $\mu$ and $\sigma$ be two fuzzy ideals of $M$ such that $\mu \subseteq \sigma$ and $\sigma(0) = \mu(0)$. Then we define a fuzzy set $\theta_{\sigma}: M/\mu \rightarrow [0, 1]$ by $\theta_{\sigma}(x + \mu) = \sigma(x)$ for all $x + \mu \in M/\mu$.

**Lemma 3.5** $\theta_{\sigma}$ is a fuzzy ideal of $M/\mu$ such that $\theta_{\mu} \subseteq \theta_{\sigma}$ where $\theta_{\sigma}$ and $\theta_{\mu}$ are given by the above notation. Also $\theta_{\mu}(0) = \theta_{\sigma}(0)$.

**Proof.** A direct verification shows that $\theta_{\sigma}$ is well-defined and is a fuzzy normal subgroup of $M/\mu$. Now we verify that $\theta_{\sigma}$ is a fuzzy ideal of $M/\mu$.

$\theta_{\sigma}((x + \mu)(y + \mu)) = \theta_{\sigma}(xoy + \mu) = \sigma(xoy)$ (by definition of $\theta_{\sigma}$) $\geq \sigma(x) = \theta_{\sigma}(x + \mu)$.

$\theta_{\sigma}\{(x + \mu)\alpha(y + \mu) - (x + \mu)\} = \theta_{\sigma}\{(x + \mu)\alpha(y + \mu) - xoy + \mu\}$

$= \theta_{\sigma}\{(x + \mu)\alpha(y + z) - xoy + \mu\}$

$= \sigma(x\alpha(y + z) - xoy)$ (by definition of $\theta_{\sigma}$)

$\geq \sigma(x) \quad \text{(since } \sigma \text{ is a fuzzy ideal of } M)$

$= \theta_{\sigma}(x + \mu) \quad \text{(by definition of } \theta_{\sigma}).$

Also $\theta_{\sigma}(x + \mu) = \sigma(x) \geq \mu(x) = \theta_{\mu}(x + \mu)$. Hence $\theta_{\mu} \subseteq \theta_{\sigma}$. \hfill $\square$

**Notation 3.6** (i) The fuzzy ideal $\theta_{\sigma}$ of $M/\mu$ is denoted by $\sigma/\mu$. Note that $\mu \subseteq \sigma$ with $\sigma(0) = \mu(0)$.

(ii) Let $\mu$ be a fuzzy ideal of $M$ and $\theta$ a fuzzy ideal of $M/\mu$ such that $\theta_{\mu} \subseteq \theta$ and $\theta_{\mu}(0) = \theta(0)$. Then we define $\sigma_{\theta}: M \rightarrow [0, 1]$ by $\sigma_{\theta}(x) = \theta(x + \mu)$ for all $x \in M$.

**Lemma 3.7** $\sigma_{\theta}$ (defined above in notation 3.6), is a fuzzy ideal of $M$ such that $\mu \subseteq \sigma_{\theta}$ and $\mu(0) = \sigma_{\theta}(0)$.

**Proof.** It is easy to verify that $\sigma_{\theta}$ is a fuzzy normal subgroup of $M$. Now $\sigma_{\theta}(xoy) = \theta(xoy + \mu) = \theta((x + \mu)\alpha(y + \mu)) \geq \theta(x + \mu)$ (since $\theta$ is a fuzzy right ideal of $M/\mu$)

$= \sigma_{\theta}(x) \quad \text{(by definition of } \sigma_{\theta}).$
Therefore $\sigma_\theta$ is a fuzzy right ideal of $M$.

Also $\sigma_\theta(x\alpha(y+z)-x\alpha y) = \theta\{(x\alpha(y+z)-x\alpha y)+\mu\}$

$$= \theta\{(x+\mu)\alpha((y+\mu)+(z+\mu))-(x+\mu)\alpha(y+\mu)\}$$

$$\geq \theta(x+\mu)$$ (since $\theta$ is a fuzzy left ideal of $M/\mu$)

$$= \sigma_\theta(x)$$ (by definition of $\sigma_\theta$).

Therefore $\sigma_\theta$ is a fuzzy left ideal of $M$.

This shows that $\sigma_\theta$ is a fuzzy ideal of $M$.

Now we have $\sigma_\theta(x) = \theta(x+\mu) \geq \theta_\mu(x+\mu)$ (since $\theta_\mu \subseteq \theta$) = $\mu(x)$ and so $\mu \subseteq \sigma_\theta$.

Also $\sigma_\theta(0) = \theta(0+\mu) = \theta(0) = \theta_\mu(0) = \mu(0)$. \qed

**Notation 3.8** Let $\mu$ be a fuzzy ideal of $M$. We write $P = \{\sigma/ \sigma$ is a fuzzy ideal of $M, \mu \subseteq \sigma, \sigma(0) = \mu(0)\}$ and $Q = \{\theta / \theta$ is a fuzzy ideal of $M/\mu, \theta_\mu \subseteq \theta$ and $\theta(0) = \theta_\mu(0)\}$.

**Theorem 3.9** Let $\mu$ be a fuzzy ideal of $M$. There exists an order preserving bijective mapping between the sets $P$ and $Q$.

**Proof.** Define $\eta: P \rightarrow Q$ by $\eta(\sigma) = \sigma$.$\sigma$. By the lemma 3.5, $\eta(\sigma) = \theta_\sigma$ is a fuzzy ideal of $M/\mu$ such that $\theta_\mu \subseteq \theta_\sigma$ and $\theta_\sigma(0) = \theta_\sigma(0)$. By the definition of $\theta_\sigma$, the mapping $\eta$ is well defined. Suppose $\eta(\sigma) = \eta(\beta) \Rightarrow \theta_\sigma = \theta_\beta$.

Now $\sigma(x) = \theta_\sigma(x+\mu) = \theta_\beta(x+\mu) = \beta(x)$ for all $x \in M$. We have proved that $\eta(\sigma) = \eta(\beta) \Rightarrow \sigma = \beta$. Therefore $\eta$ is one-one.

Let $\theta \in Q$. Consider $\sigma_\theta: M \rightarrow [0, 1]$ defined in notation 3.6 (ii). By Lemma 3.7, $\sigma_\theta \in P$. Now we have to show that $\eta(\sigma_\theta) = \theta$.

$(\eta(\sigma_\theta))(x+\mu) = \sigma_\theta(x)$ (by definition of $\eta$ and the definition of $\theta_\sigma$ in notation 3.4) = $\theta(x+\mu)$ (by the definition of $\sigma_\theta$ in notation 3.6 (ii)).

This is true for all $x+\mu \in M/\mu$. Hence $\eta(\sigma_\theta) = \theta$ and so $\eta$ is onto.

Thus $\eta: P \rightarrow Q$ is a bijection.

Let $\sigma, \beta \in P$ such that $\sigma \subseteq \beta$. Now $(\eta(\sigma))(x+\mu) = \theta_\sigma(x+\mu)$ (by the definition of $\eta$)

$\sigma(x) \leq \beta(x)$ (since $\sigma \subseteq \beta$) = $\theta_\beta(x+\mu) = (\eta(\beta))(x+\mu)$.

Since this is true for all $x+\mu \in M/\mu$, we have that $\eta(\sigma) \subseteq \eta(\beta)$.

Thus $\eta: P \rightarrow Q$ is an order preserving bijection. \qed
Theorem 3.10 (Jun, Sapanci and Ozturk [7]) A Γ-nearing homomorphic pre-image of a fuzzy (left, right) ideal is a fuzzy (left, right) ideal.

Theorem 3.11 Let $h: M \rightarrow M^1$ be an epimorphism, $\sigma$ is a fuzzy ideal of $M^1$ and $\mu = h^{-1}(\sigma)$. Then the map $\psi: M/\mu \rightarrow M^1/\sigma$ defined by $\psi(x+\mu) = h(x) + \sigma$ is a Γ-near-ring isomorphism.

Proof. First we show that the mapping $\psi$ is well defined.

Let $z^1 \in M^1$. Since $h$ is an epimorphism, $h(z) = z^1$ for some $z \in M$.

Now $x+\mu = y+\mu \Rightarrow (x+\mu)(z) = (y+\mu)(z) \Rightarrow \mu(x-z) = \mu(y-z) \Rightarrow (h^{-1}(\sigma))(x-z) = (h^{-1}(\sigma))(y-z) \Rightarrow \sigma(h(x-z)) = \sigma(h(y-z)) \Rightarrow \sigma(h(x)-z^1)) = \sigma(h(y)-z^1))$

$\Rightarrow (h(x)+\sigma)(z^1) = (h(y)+\sigma)(z^1)$

This is true for all $z^1 \in M^1$. Hence $h(x) + \sigma = h(y) + \sigma$.

Now we proved that $x+\mu = y+\mu \Rightarrow \psi(x+\mu) = \psi(y+\mu)$. Thus $\psi$ is well defined.

It is easy to verify that $\psi((x+\mu)+(y+\mu)) = \psi(x+\mu) + \psi(y+\mu)$.

Now $\psi((x+\mu)\sigma(y+\mu)) = \psi(x\sigma y + \mu) = h(x\sigma y) + \sigma$, by definition of $\psi$. Since $h$ is a homomorphism, we have $h(x\sigma y) + \sigma = (h(x)\sigma h(y)) + \sigma = (h(x)+\sigma)(h(y)+\sigma)$

$= \psi(x+\mu)\sigma\psi(y+\mu)$, by definition of $\psi$. Therefore $\psi$ is a homomorphism.

Now we verify that $\psi$ is one-one. Suppose $h(x)+\sigma = h(y)+\sigma$. Then $\sigma[h(x)-h(y)] = \sigma[h(0)]$, by definition. Since $h$ is a homomorphism, $\sigma[h(x-y)] = \sigma(h(0))$. This implies $(h^{-1}(\sigma))(x-y) = (h^{-1}(\sigma))(0)$, which implies $\mu(x-y) = \mu(0)$. By Lemma 2.2 (i), $x+\mu = y+\mu$. This shows that $\psi$ is one-one. Let $y \in M^1/\mu$. Then $y = h(x)+\sigma$ for some $x \in M$. Now $\psi(x + \mu) = h(x) + \sigma = y$. Therefore $\psi$ is onto. Hence $\psi$ is an isomorphism. \(\square\)

As a consequence of Theorem 3.11, we obtain the following corollary.

Corollary 3.12 Let $\mu$ and $\sigma$ be two fuzzy ideals of $M$ such that $\mu \subseteq \sigma$ and $\sigma(0) = \mu(0)$. Then $M/\sigma \cong (M/\mu)/(\sigma/\mu)$.

Proof. Define $\psi: M \rightarrow M/\mu$ by $\psi(x) = x + \mu$ for all $x \in M$.

By theorem 3.3, $\psi$ is Γ-nearing epimorphism. From the notation 3.4 and 3.6 we have $\theta_\sigma = \sigma/\mu$ and by Lemma 3.5, $\sigma/\mu$ is a fuzzy ideal of $M/\mu$ such that $\theta_\mu \subseteq \theta_\sigma = \sigma/\mu$ and $\theta_\mu(0) = \theta_\sigma(0)$. Now $\psi^{-1}(\sigma/\mu)$ is a fuzzy set in $M$ and for any $x \in M$ we have

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\[(\psi^{-1}(\sigma/\mu))(x) = \psi^{-1}(\theta(x)) = \theta(x+\mu) = \sigma(x)\]. Therefore \(\psi^{-1}(\sigma/\mu) = \sigma\) is a fuzzy ideal of \(M\). Define \(\psi^* : M/\sigma \rightarrow (M/\mu)/(\sigma/\mu)\) by \(\psi^*(x+\sigma) = \psi(x) + (\sigma/\mu)\).

By theorem 3.11, \(\psi^*\) is a \(\Gamma\)- nearring isomorphism. This completes the proof. \(\Box\)

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