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RIESZ THEOREMS IN 2-INNER PRODUCT SPACES

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Abstract. In this paper we describe the proof of 'Riesz Theorems' in 2-inner product spaces. The main result holds only for a b-linear functional but not for a bilinear functional.

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1. Introduction

The concepts of 2-inner product and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [1].

2. Preliminaries

Definition 2.1. ([4]) Let X be a real linear space of dimension greater than 1 and $\|.,.\|$ be a real valued function on X×X satisfying the properties,

A1 : $\|x, y\|$=0 iff vectors x and y are linearly dependent.
A2 : $\|x, y\| = \|y, x\|$
A3 : $\|x, \alpha y\| = |\alpha| \|x, y\|$
A4 : $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for every $x, y, z \in X$ and $\alpha \in R$

then the function $\|.,.\|$ is called a 2-norm on X. The pair $(X, \|.,.\|)$ called linear 2-normed space.

Every 2-normed space is a locally convex TVS. In fact, for a fixed $b \in X, P_b(x) = \|x, b\|, x \in X$ is a seminorm and the family $\{P_b; b \in X\}$ of seminorms generates a locally convex topology on X.

Definition 2.2. ([4]) Let $(X, \|.,.\|)$ be a 2-normed space and $x, y \in X$ then x is said to be b-orthogonal to y iff there exists $b \in X$ such that for every α, $\|x, b\| \neq 0$, $\|x, b\| \leq \|x + \alpha y, b\|$ and $y \neq b$.

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Definition 2.3. ([4]) Let X be a linear space of dimension greater than 1 over the field K (either R or C). The function $\langle .,.;.\rangle : X \times X \times X \to K$ is called a 2-inner product if the following conditions holds,

A1 :$\langle x, x; z\rangle \geq 0$ and $\langle x, x; z\rangle = 0$ iff x and z are linearly dependent.
A2 :$\langle x, x; z\rangle = \langle z, z; x\rangle$.
A3 :$\langle x, y; z\rangle = \langle y, x; z\rangle$.
A4 :$\langle \alpha x, y; z\rangle = \alpha \langle x, y; z\rangle$, for all scalars $\alpha \in K$.
A5 :$\langle x_1 + x_2, y; z\rangle =\langle x_1, y; z\rangle + \langle x_2, y; z\rangle$.

Therefore, the pair $(X, \langle .,.;.\rangle)$ is called a 2-inner product space.

Let $(X, \langle .,.;.\rangle)$ be a 2-inner product space and $x, y, b \in X$ then $x \perp^b y$ iff $\langle x, y; b\rangle = 0$ [5].

We define a 2-norm on $X \times X$ by,

$$\|x, y\|^2 = \langle x, x; y\rangle .$$

Definition 2.4. ([3]) Let $(X, \langle .,.;.\rangle)$ be a 2-inner product space over K. If $\{e_i\}_{1\leq i\leq n}$ are linearly independent vectors in the 2-inner product space X, then $\{e_i\}_{1\leq i\leq n}$ is called a b-orthonormal set if for $b \in X$, $\langle e_i, e_j; b\rangle = 0$ if $i \neq j$ and $\langle e_i, e_j; b\rangle = 1$ if i = j where $1 \leq i \leq n$.

Definition 2.5. ([4]) Let $(X, \langle .,.;.\rangle)$ be a 2-inner product space over K, $b \in X$, then

(a) A sequence $\{x_n\}$ in X is said to be a b-Cauchy sequence if for every $\epsilon > 0$ there exists $N > 0$ such that for every $m, n \geq N$, $0 < \|x_n - x_m, b\| < \epsilon$.

(b) X is said to be b-Hilbert if every b-Cauchy sequence is convergent in the semi-normed space $(X, \|., b\|)$.

Theorem 2.6. *Let $\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n\}$ be a linearly independent subset of a 2-inner product space $(X, \langle .,.;.\rangle)$. For $b \in X$ there exists a b-orthonormal set $\{e_1, e_2, e_3, ..., e_n\}$ in X such that* $\operatorname{span}\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n\} = \operatorname{span}\{e_1, e_2, e_3, ..., e_n\}$.

Theorem 2.7 (Bessel's Inequality in 2-inner product spaces ([2]))**.** *Let $(X, \langle .,.;.\rangle)$ be a 2-inner product space over the scalar field K, then*

$$\sum_{i=1,2...n} \left|\langle x, e_i; b\rangle^2\right| \leq \|x, b\|^2$$

which holds for any $x \in X$ whenever $e_1, e_2, e_3, ..., e_n$, $b \in X$ are the vectors such that $b \in \operatorname{span}\{e_1, e_2, e_3, ..., e_n\}$ and $\langle e_i, e_j; b\rangle = 0$ if $i \neq j$ and $\langle e_i, e_j; b\rangle = 1$ if $i = j$ where $1 \leq i \leq n$. Also, the equality holds iff $x = u + \gamma b$ for some $u \in \operatorname{span}\{e_1, e_2, e_3, ..., e_n\}$ and some $\gamma \in K$.

Theorem 2.8 (Cauchy Schwartz Inequality) [1, 2, 3])**.** *Let $(X, \langle .,.;.\rangle)$ be a 2-inner product space over the scalar field K, then*

$$|\langle x, y; z\rangle| \leq \|x, z\| \, \|y, z\|$$

for every $x, y, z \in X$

Theorem 2.9. *Let $\{e_\alpha\}$ be a b-orthonormal set in a 2-inner product space X and $x, b \in X$, then $E_x = \{e_\alpha; \langle x, e_\alpha; b\rangle = 0\}$ is countable.*

3. Main Results

Throughout this section we assume that X is a vector space of dimension greater than 1.

Definition 3.1. Let $(X, \|.,.\|)$ be a 2-normed space. Let W be a subspace of X, $b \in X$ be fixed, then a map $T : W \times \langle b \rangle \to K$ is called a b-linear functional on $W \times \langle b \rangle$ whenever for every $x, y \in W$ and $k \in K$ holds

1. T(x+y,b)= T(x, b) + T(y, b),
2. T(k x , b) = k T(x , b).

A b-linear functional $T : W \times \langle b \rangle \to K$ is said to be bounded if there exists a real number $M > 0$ such that $|T(x, b)| \leq M \|x, .b\|$ for every $x \in W$.

The norm of the b-linear functional $T : W \times \langle b \rangle \to K$ is defined by

$$\|T\| = inf \{M > 0; |T(x, b)| \leq M \|x, .b\|, \forall x \in W\}$$

It can be seen that,

$$\|T\| = \sup \{|T(x, b)|; \|x, .b\| \leq 1\}$$

$$\|T\| = \sup \{|T(x, b)|; \|x, .b\| = 1\}$$

$$\|T\| = \sup \{|T(x, b)| / \|x, .b\|; \|x, .b\| \neq 0\}$$

and $|T(x, b)| \leq \|T\| \|x, .b\|$

For a 2-normed space $(X, \|.,.\|)$ and $0 \neq b \in X$, X_b^* denote the Banach space of all bounded b-linear functionals on $X \times \langle b \rangle$, where $\langle b \rangle$ is the subspace of X generated by 'b'.

Theorem 3.2. *Let $(X, \langle .,.;. \rangle\ )$ be a 2-inner product space and $\{e_1, e_2, e_3, ...\}$ be a b-orthonormal set in X and $k_1, k_2, k_3, ... \in K$ then,*

(i) If $\sum_n k_n e_n$ converges to some x in the semi-normed space $(X, \|., b\|)$, then $\langle x, e_i; b \rangle = k_n$ for each n and $\sum_n |k_n|^2 < \infty$.

(ii) If X is a b-Hilbert space and $\sum_n |k_n|^2 < \infty$ then $\sum_n k_n e_n$ converges to some x in the semi-normed space $(X, \|., b\|\ )$.

Proof. (i) If $\sum_n k_n e_n$ converges to some x in X, then $x = \sum_n k_n e_n$. Since $\{e_1, e_2, e_3, \ldots\}$ is a b-orthonormal set in X, we get $\langle x, e_i; b \rangle = k_i$ for each i. Therefore, by Theorem 2.8, $\sum_n |k_n|^2 = \|x, .b\|^2 < \infty$.

(ii) For $m = 1, 2, 3, \ldots$, let $x_m = \sum_{n=1}^{m} k_n e_n$.

Therefore, $m > j, x_m - x_j = \sum_{n=j+1}^{m} k_n e_n$.

We have $\|x_m - x_j, .b\|^2 = \langle x_m - x_j, x_m - x_j; b \rangle = \sum_{n=j+1}^{m} |k_n|^2 < \infty$.

Therefore, $\{x_m\}$ is a b-Cauchy sequence in $(X, \|., b\|)$. Since X is a b-Hilbert space, $\{x_m\}$ converges to some x in X. □

Theorem 3.3. *Let $\{e_\alpha\}$ be a b-orthonormal basis in a b-Hilbert space X, then for every x in X, $x = \sum_n \langle x, e_n; b \rangle e_n$.*

Proof. Since $\{e_\alpha\}$ is a b-orthonormal basis in a 2-inner product space X, $\{e_\alpha\}$ is a countable set, say $\{e_1, e_2, e_3, ...\}$.

By Theorem 2.8, we have, $\sum |\langle x, e_n; b\rangle|^2 \leq \|x, .b\|^2 < \infty$, $\Rightarrow$ $|\langle x, e_n; b\rangle|^2$ converges to 0 as $n \to \infty$.

Therefore, by Theorem 3.2(ii), $\sum_n \langle x, e_n; b\rangle e_n$ converges to some y in X.

That is, $y = \sum_n \langle x, e_n; b\rangle e_n$.

Also, $\langle y, e_n; b\rangle = \langle \sum_n \langle x, e_i; b\rangle e_i, e_n, ; b\rangle = \langle x, e_n; b\rangle$.

This implies $\langle x - y, e_n; b\rangle = 0$. So, $(x - y)\perp^b e_n$ for all n.

If $y \neq x$ then let $u = (x - y)/\, \|x - y, b\| \Rightarrow \|u, b\| = 1$. Since $(x - y)\perp^b e_n$ for all n, $\langle u, e_n; b\rangle = 0$. Therefore, $\{e_n\} \bigcup \{u\}$ is a b-orthonormal set in X, which contradicts the maximality of the b-orthonormal set $\{e_\alpha\}$. So, $y = x$. Hence, $x = \sum_n \langle x, e_n; b\rangle e_n$. □

Definition 3.4. Let X be a vector space over K. Let $b \in X$ and $y_1, y_2 \in X$, then y_1 is said to be b-congruent to y_2 iff $(y_1 - y_2) \in \langle b\rangle$ is the subspace generated by b.

Theorem 3.5. *Let X be a b-Hilbert space and $T \in X_b^*$ then there exists a unique $y \in X$ up to b-congruence such that $T(x, b) = \langle x, y; b\rangle$ and $\|T\| = \|y, .b\|$.*

Proof. Let $\{e_1, e_2, e_3, \ldots\}$ be a b-orthonormal set.

For $m = 1, 2, 3, \ldots$ let $y_m = \sum_{n=1}^m T(e_n, b) e_n$.

Since $\{e_1, e_2, e_3, ...\}$ is a b-orthonormal set,

$$\|y_m, .b\|^2 = \textstyle\sum_{n=1}^m |T(e_n, b)|^2 = \beta_m.$$

Also, $T(y_m, b) = \sum_{n=1}^m |T(e_n, b)|^2 = \beta_m$.

Since T is bounded, $|T(e_n, b)| \leq \|y_m, .b\| \Rightarrow \beta_m \leq \|T\|^2$.

Letting $m \to \infty$, $\sum_n |T(e_n, b)|^2 \leq \|T\|^2 < \infty$.

Let $\{e_\alpha\}$ be a b-orthonormal basis for X. Set $E_T = \{\{e_\alpha\}; T(e_\alpha, b) \neq 0\}$ is countable and let $E_T = \{e_1, e_2, e_3, ...\}$. Then $\sum_n |T(e_n, b)|^2 < \infty$. Therefore, by Theorem 3.2(ii), $\sum_n \overline{T(e_n, b)} e_n$ converges in X.

Let y $= \sum_n \overline{T(e_n, b)} e_n$.

Claim: T(x, b) $= \langle x, y; b\rangle$ for every x in X.

Let $x \in X$, then $\{e_\alpha; \langle x, e_\alpha; b\rangle \neq 0\}$ is countable (see Theorem 2.9). Let it be $\{s_1, s_2, s_3, ...\}$. Then $x = \sum_m \langle x, s_m; b\rangle s_m \Rightarrow T(x, b) = \sum_m \langle x, s_m; b\rangle T(s_m, b)$. To prove the claim it is sufficient to show that $T(x, b) = \langle s_m, y; b\rangle$ for $m = 1, 2, 3, \ldots$. Fix m and let $\langle s_m, y; b\rangle = \sum_n T(e_n, b) \langle s_m, e_n; b\rangle$. If $s_m = e_{n_\circ}$ for some $n_\circ$, then $\langle s_m, y; b\rangle = T(e - n_\circ, b) = T(s_m, b)$. If $s_m \neq e_n$ for some n, then $\langle s_m, y; b\rangle = 0$, implying $T(s_m, b) = 0$. Therefore, $T(s_m, b) = \langle x, y; b\rangle$ for all m. Hence $T(x, b) = \langle x, y; b\rangle$.

Let us prove the uniqueness of such y.

Let $y_1, y_2 \in X$ such that $T(x, b) = \langle x, y_1; b\rangle$ and $T(x, b) = \langle x, y_2; b\rangle$. This gives $\langle x, y_1; b\rangle = \langle x, y_2; b\rangle$, which implies $\langle x, y_1 - y_2; b\rangle = 0$ for all x in X.

In particular, $\langle y_1 - y_2, y_1 - y_2; b\rangle = 0$, so $y_1 - y_2 = kb$ for some $k \in K$, implying $y_1 - y_2 \in \langle b\rangle$

Therefore y is unique up to b-congruence.

It can be easily shown that $\|T\| = \|y, b\|$.

If $T = 0$, then $T(x, b) = 0$ for every x. Also $\Rightarrow \langle x, y; b\rangle = 0$ for every x, so y and b are linearly dependent $\Rightarrow \|y, b\| = 0$.

Therefore, $\|y, b\| = 0 = \|0, b\| = \|T\|$.

If $T \neq 0$, then $T(x, b) \neq 0$ for all x, which gives $\langle x, y; b\rangle \neq 0$ for every x.

So, $y \neq 0$ or y and b are linearly independent.

Therefore, $\|y, b\|^2 = \langle y, y; b\rangle = T(y, b) \leq \|T\| \, \|y, b\|$.

So,

$$\|y, b\| \leq \|T\| \tag{1}$$

and, by Cauchy Schwartz Inequality, $T(x, b) = |\langle x, y; b\rangle| \leq \|x, b\| \, \|y, b\|$, which gives

$$\|T\| = \sup \{|T(x, b)| \, ; \|x, .b\| = 1\} = \sup |\langle x, y; b\rangle| \leq \|y, b\| \, . \tag{2}$$

From (1) and (2) we get $\|T\| = \|y, b\|$.

Hence the theorem. □

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References

[1] Cho, Y. J., Lin, P. C. S., Kim, S. S., Misiak. A., Theory of 2-inner product spaces. New York: Nova Science Publishes, Inc. 2001.

[2] Cho, Y. J., Matic, M., Pecaric, J. E., On Gram's determinant in 2-inner product spaces. J. Korean Math. Soc. 38 (6) (2001), 1125-1156.

[3] Dragomir, S. S., Cho, Y. J., Kim, S. S., Sofo, A., Some Boas - Bellman type Inequalities in 2-inner product spaces. J. Inequal. in Pure and Appl. Math. 6(2)55 (2005), 1-13.

[4] Kazemi, R., Mazaheri, H., Some results on 2-inner product spaces. Novi Sad J. Math. Vol. 37. No. 2, (2007), 35-40.

[5] Mazaheri. H., Golestani Nezhad , Some results on b-orthogonality in 2-normed spaces. Int. Journal of Math. Analysis, Vol. 1, No. 14. (2007), 681-687.

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