Some Properties of Accretive Operators in Linear 2-Normed Spaces

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Abstract
In this paper we discuss some properties of resolvents of an accretive operator in linear 2-normed spaces, focusing on the concept of contraction mapping and the unique fixed point of contraction mappings in linear 2-normed spaces. Also, we establish the existence of solution of strong accretive operator equation in linear 2-normed spaces.

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1 Introduction
The concept of 2-metric spaces, linear 2-normed spaces and 2-inner product spaces, introduced by S Gahler in 1963, paved the way for a number of authors like, A White, Y J Cho, R Freese, C R Diminnie, to do work on possible applications of Metric geometry, Functional Analysis and Topology as a new tool. A systematic presentation of the recent results related to the Geometry
of linear 2-normed spaces as well as an extensive list of the related references can be found in the book [1]. In [4] S Gahler introduced the following definition of linear 2-normed spaces. The study of accretive operators in various spaces has been done by the authors T Kato[6], Shih-sen Chang, Yeol Je Cho, Shin Min Kang [5].

2 Preliminaries

Definition 2.1 (4) Let X be a real linear space of dimension greater than 1 and \( \|.,.\| \) be a real valued function on \( X \times X \) satisfying the properties,
\[ A1: \|x,y\| = 0 \text{ iff } x \text{ and } y \text{ are linearly dependent} \]
\[ A2: \|x,y\| = \|y,x\| \]
\[ A3: \|\alpha x,y\| = |\alpha| \|y,x\| \]
\[ A4: \|x + y,z\| \leq \|x,z\| + \|y,z\| \]
then the function \( \|.,.\| \) is called a 2-norm on X. The pair \((X,\|.,.\|)\) is called a linear 2-normed space.

Some of the basic properties of 2-norms, they are non-negative and \( \|x,y + \alpha x\| = \|x,y\| \) \( \forall x,y \in X \) and \( \alpha \in \mathbb{R} \).

The most standard example for a linear 2-normed space is \( X = \mathbb{R}^2 \) equipped with the following 2-norm,
\[ \|x_1,x_2\| = \text{abs det}(x_{11} x_{12} x_{21} x_{22}) \] where \( x_i = (x_{i1}, x_{i2}) \) for \( i=1,2 \)

Every linear 2-normed space is a locally convex TVS. In fact, for a fixed \( b \in X \), \( P_b(x) = \|x,b\| \) for \( x \in X \) is a semi norm and the family \( \{P_b; b \in X\} \) of semi norms generates a locally convex topology on X.

Definition 2.2 (1) Let \((X,\|.,.\|)\) be a linear 2-normed space, \( E \) be a subset of \( X \) then the closure of \( E \) is \( \overline{E} = \{x \in X; \text{ there is a sequence } x_n \text{ of } E \text{ such that } x_n \to x\} \). We say, \( E \) is sequentially closed if \( E = \overline{E} \).

Definition 2.3 (1) Let \((X,\|.,.\|)\) be a linear 2-normed space, then the mapping \( T: X \to X \) is said to be sequentially continuous if \( x_n \to x \) implies \(Tx_n \to Tx\).

Definition 2.4 (3) An operator \( T: D(T) \subset X \to X \) is said to be accretive if for every \( z \in D(T) \)
\[ \|x - y,z\| \leq \|(x - y) + \lambda(Tx - Ty),z\| \] for all \( x,y \in D(T) \) and \( \lambda > 0 \).
Definition 2.5 (3) An operator \( T : D(T) \subset X \rightarrow X \) is said to be \( m \)-accretive if \( R(I + \lambda T) = X \) for \( \lambda > 0 \).

Remark 2.6 (4) Every closed subset of a 2-Banach space is complete.

3 Main Results

Consider an accretive operator \( T : D(T) \subset X \rightarrow X \) then for \( \lambda > 0 \), \((I + \lambda T)^{-1}\) exists. We have \( T : D(T) \rightarrow R(T) \) then \((I + \lambda T)\) is onto. Let \( x_1, x_2 \in D(T) \) with \((I + \lambda T)x_1 = (I + \lambda T)x_2 \) implies \((x_1 - x_2) + \lambda(Tx_1 - Tx_2) = 0 \) implies \( \|(x_1 - x_2) + \lambda(Tx_1 - Tx_2), z\| = 0 \) for every \( z \in X \). Since \( T \) is accretive, we have \( \|x_1 - x_2, z\| \leq 0 \) for every \( z \in X \) implies \( \|x_1 - x_2, z\| = 0 \) for every \( z \in X \) implies \( x_1 - x_2 = 0 \) implies \( x_1 = x_2 \). So \((I + \lambda T)\) is one-one. Hence for \( \lambda > 0 \), \((I + \lambda T)^{-1}\) exists.

Definition 3.1 Let \((X, \|\cdot,\cdot\|)\) be a linear 2-normed space then an operator \( T \) on \( X \) is said to be non expansive if for each \( x,y \in D(T) \), \( \|Tx - Ty, z\| \leq \|x - y, z\| \) for every \( z \in X \). \( T \) is said to be expansive if for each \( x,y \in D(T) \), \( \|Tx - Ty, z\| > \|x - y, z\| \) for every \( z \in X \).

Proposition 3.2 Let \((X, \|\cdot,\cdot\|)\) be a linear 2-normed space and \( T : D(T) \subset X \rightarrow X \) be an operator. If \((I + \lambda T)^{-1}\) is expansive for all \( \lambda > 0 \) then \( T \) is accretive.

Proof: The proof is immediate from the definition 3.1.

Remark 3.3 If \( T \) is an accretive operator on a 2-normed space \( X \) then \((I + \lambda T)^{-1}\) is non expansive.

Definition 3.4 Let \((X, \|\cdot,\cdot\|)\) be a linear 2-normed space \( E \) be a non empty subset of \( X \) and \( e \in E \) then \( E \) is said to be \( e \)-bounded if there exists some \( M > 0 \) such that \( \|x, e\| \leq M \) for all \( x \in E \). If for all \( e \in E \), \( E \) is \( e \)-bounded then \( E \) is called a bounded set.

Let \((X, \|\cdot,\cdot\|)\) be a linear 2-normed space and \( T \) be an \( m \)-accretive operator on \( X \). For \( n = 1, 2, 3, \ldots \), Define the resolvent of \( T \) as, \( J_n(x) = (I + n^{-1}T)^{-1}(x) \) and the Yosida approximation \( T_n(x) = n(I - J_n)(x) \) for all \( x \in X \) and \( \lambda > 0 \).

Proposition 3.5 Let \((X, \|\cdot,\cdot\|)\) be a linear 2-normed space and \( T \) be a \( m \)-operator then

(i) \( \|J_n x - J_n y, z\| \leq \|(x - y), z\| \)

(ii) \( \|T_n x - T_n y, z\| \leq 2n \|(x - y), z\| \) for all \( x, y \in X \)
Proof:
(i) Since $T$ is accretive in $X$, we have $(I + \alpha T)^{-1}$ is non expansive implies for each $x,y \in D(T)$, $||(I + \alpha T)^{-1} x - (I + \alpha T)^{-1} y, z|| \leq ||x - y, z||$
for every $z \in X$
Take $\alpha = 1/n$ then $\alpha > 0$. So $\|J_n x - J_n y, z\| \leq \|(x - y), z\|$.
(ii) We have for each $x,y \in D(T)$,
$$
\|T_n x - T_n y, z\| = \|n(I - J_n)x - (I - J_n)y, z\|
= \|n(x - y) + n(J_n x - J_n y), z\|
\leq n\|x - y, z\| + n\|J_n x - J_n y, z\|
\leq n\|x - y, z\| + n\|x - y, z\|
= 2n\|x - y, z\|
$$
for all $z \in X$

**Proposition 3.6** Let $(X, \|\cdot\|)$ be a linear 2-normed space and $E$ be a non empty bounded subset of $X$ and $T : E \to E$ be an accretive operator then there exists some $M > 0$ and for $x \in E$, $\|T_n x, z\| \leq M$ for all $z \in E$.

Proof: Let $x \in E$ then
$$
T_n x = n(x - J_n x) = n(J_n(J_n^{-1} x) - J_n x) = n(J_n(I + n^{-1} T)x - J_n x)
$$
So, by Proposition 3.5 (i);
$$
\|T_n x, z\| = \|n(J_n(I + n^{-1} T)x - J_n x), z\| \leq n\|(I + n^{-1} T)x - x, z\|
= \|T x, z\| \leq M
$$
for all $z \in E$

**Proposition 3.7** Let $(X, \|\cdot\|)$ be a linear 2-normed space and $E$ be a non empty bounded subset of $X$ and $T : E \to E$ be an accretive operator. If $x \in E$ then $J_n x \to x$.

Proof: Let $x \in E$ then $\|x - J_n x, z\| = n^{-1}\|T_n x, z\| \leq n^{-1}M$ for some $M > 0$
and for all $z \in E$.

ie; $\|x - J_n x, z\| \to 0$ as $n \to \infty$ implies $J_n x \to x$
If $x \in E$ then there exists a sequence $\{x_m\} \in E$ such that $x_m \to x$. Since $x_m \in E$ we have $J_n(x_m) \to x_m$.

We have, for all $z \in E$
$$
\|J_n x - x, z\| = \|J_n x - J_n(x_m) + J_n(x_m) - J_n(x_m) + J_n(x_m) - x_m + x_m - x, z\|
\leq \|J_n x - J_n(x_m), z\| + \|J_n(x_m) - x_m, z\| + \|x_m - x, z\|
\leq \|x - x_m, z\| + \|J_n(x_m) - x_m, z\| + \|x_m - x, z\| \to 0\text{ as } n \to \infty
$$

Hence $J_n x \to x$. 
Definition 3.8 Let $(X, ||.||)$ be a linear 2- normed space then the mapping $T : X \rightarrow X$ is said to be a contraction if there exists some $k \in (0, 1)$ such that $\|Tx - Ty, z\| \leq k\|x - y, z\|$ for all $x, y, z \in X$.

Lemma 3.9 Let $(X, ||.||)$ be a linear 2- normed space then every contraction $T : X \rightarrow X$ is sequentially continuous.

Proof: Since $T$ is a contraction, there exists some $k \in (0, 1)$ such that $\|Tx - Ty, z\| \leq k\|x - y, z\|$ for all $x, y, z \in X$.

Let $\{x_n\}$ be a sequence in $X$ such that $x_n \rightarrow x$.

Then, $\|Tx_n - Tx, z\| \leq k\|x_n - x, z\|$ as $n \rightarrow \infty$ implies $Tx_n \rightarrow Tx$.

So $T$ is sequentially continuous.

Next lemma proves the analogues of Banach fixed point theorem in Linear 2-normed spaces.

Lemma 3.10 Let $(X, ||.||)$ be a linear 2- normed space and $E$ be a non empty closed and bounded subset of $X$. Let $T : E \rightarrow E$ be a contraction then $T$ has a unique fixed point on $X$.

Proof: Since $T$ is a contraction, there exists some $k \in (0, 1)$ such that

$\|Tx - Ty, z\| \leq k\|x - y, z\|$ for all $x, y, z \in X$.

We have, $\|T^2x - T^2y, z\| = \|T(Tx) - T(Ty), z\| \leq k\|Tx - Ty, z\| \leq k^2\|x - y, z\|$ for all $z \in E$. Similarly $\|T^n x - T^n y, z\| \leq k^n\|x - y, z\|$ for all $z \in E$.

Let $x_0 \in E$ then construct a sequence depending on $x_0$.

Let $x_1 = Tx_0$, $x_2 = Tx_1$, $x_3 = Tx_2$, ..., $x_n = Tx_{n-1}$ then $x_1 = Tx_0$, $x_2 = T^2x_0$, $x_3 = T^3x_0$, ..., $x_n = T^n x_0$.

We show that $\{x_n\}$ is a cauchy sequence in $E$. Let $m, n > 0$ with $m > n$. Take $m = n + p$ then for any $z \in E$,

$\|x_n - x_m, z\| = \|x_n - x_{n+p}, z\|
= \|(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + x_{n+2} - ... + (x_{n+p-1} - x_{n+p}), z\|
\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + ... + \|x_{n+p-1} - x_{n+p}, z\|
= \|T^n x_0 - T^{n+1} x_1, z\| + \|T^{n+1} x_0 - T^{n+2} x_1, z\| + ... + \|T^{n+p-1} x_0 - T^{n+p} x_1, z\|
\leq k^n \|x_0 - x_1, z\| + k^{n+1} \|x_0 - x_1, z\| + ... + k^{n+p-1} \|x_0 - x_1, z\|
\leq k^n \|x_0 - x_1, z\|(1 + k + k^2 + ...) = (\frac{k^n}{1-k}) \|x_0 - x_1, z\|
1e; $\|x_n - x_m, z\| \leq (\frac{k^n}{1-k}) \|x_0 - x_1, z\|$ for all $z \in E$.

Since $E$ is bounded, there exists $M > 0$ such that $\|x_0 - x_1, z\| \leq M$ for all $z \in E$. 
So, \( \|x_n - x_m, z\| \leq \left( \frac{k + M}{1 - k} \right) \) for all \( z \in E \) implies \( \|x_n - x_m, z\| \rightarrow 0 \) as \( n \rightarrow \infty \), since \( k \in (0, 1) \) implies \( \{x_n\} \) is a cauchy sequence in \( E \), So \( \{x_n\} \) converges to some \( x \) in \( E \).

Since \( T \) is sequentially continuous, \( Tx = \lim Tx_n = \lim x_{n+1} = x \) as \( n \rightarrow \infty \). Therefore \( T \) has a fixed point in \( E \).

Now, we have to prove that such a fixed point is unique.

Let \( y \in E \) with \( y \neq x \) and \( Ty \neq y \).

Assume that \( Ty = y \) then \( \|x - y, z\| = \|Tx - Ty, z\| \leq k\|x - y, z\| \) implies \( k \geq 1 \), a contradiction to \( k \in (0, 1) \). So, \( Ty \neq y \). Hence the fixed point of \( T \) is unique in \( E \).

**Definition 3.11** Let \( (X, \|\cdot, \cdot\|) \) be a linear 2- normed space then the mapping \( T : X \rightarrow X \) is said to be strong accretive if for every \( z \in X \)

\[
\|(\lambda - k)(x - y), z\| \leq \|(\lambda - 1)(x - y) + (Tx - Ty), z\|
\]

for all \( x, y \in X \) and \( \lambda > k \), and \( k \in (0, 1) \).

**Theorem 3.12** Let \( (X, \|\cdot, \cdot\|) \) be a linear 2- normed space and \( E \) be a non empty closed and bounded subset of \( X \). Let \( T : E \rightarrow X \) be a strong accretive mapping. If \( (I + T)(E) \supset E \) then the equation \( Tx = \theta \) has a solution in \( E \), where \( \theta \) is the zero vector in \( X \).

Proof: Since \( T \) is strong accretive implies

\[
\|(\lambda - k)(x - y), z\| \leq \|(\lambda - 1)(x - y) + (Tx - Ty), z\|
\]

for all \( x, y \in X \) and \( \lambda > k \), and \( k \in (0, 1) \)

Put \( \lambda = 2 \) then \( \|(2 - k)(x - y), z\| \leq \|(x - y) + (Tx - Ty), z\| \)

implies \( \|2 - k\|(x - y), z\| \leq \|(I + T)(x) - (I + T)(y), z\| \)

implies \( \|(x - y), z\| \leq \frac{\|(I + T)(x) - (I + T)(y), z\|}{|2 - k|} \)

Since \( (I + T)(E) \supset E \) we get,

\( x, y \in D[(I + T)^{-1}] \) implies \( x = (I + T)^{-1}(u) \) and \( y = (I + T)^{-1}(v) \) for some \( u, v \in E \)

So, \( \|(I + T)^{-1}(u) - (I + T)^{-1}(v), z\| \leq \frac{\|u - v, z\|}{|2 - k|} \)

ie; \( (I + T)^{-1} : E \rightarrow E \) is a contraction mapping. By the Lemma 3.10 \( (I + T)^{-1} \) has a fixed point \( x_0 \) in \( E \). ie; \( x_0 = (I + T)^{-1}(x_0) \) implies \( x_0 = (I + T)(x_0) \) implies \( Tx_0 = \theta \).

Hence, the equation \( Tx = \theta \) has a solution in \( E \).

**Corollary 3.13** Let \( (X, \|\cdot, \cdot\|) \) be a linear 2- normed space and \( E \) be a non empty closed and bounded subset of \( X \). Let \( T : E \rightarrow E \) be a strong accretive mapping. If \( (I + T)(E) = E \) then \( R(T) = E \).
Proof:
For any \( p \in E \). Take \( T_0 = T - p \)
Since \( T \) is strong accretive we have,
\[
\| (\lambda - k)(x - y), z \| \leq \| (\lambda - 1)(x - y) + (Tx - Ty), z \| \text{ for all } x, y \in X \text{ and } \lambda > k, \text{ and } k \in (0, 1)
\]
We have, \( T_0 \) is also strong accretive.
Therefore, by theorem 3.12, there exists \( x_0 \in E \) such that \( T_0(x_0) = \theta \)
implies \( Tx_0 - p = \theta \) implies \( Tx_0 = p \) for every \( p \in E \). Therefore, \( R(T) = E \).

References


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