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Some Properties of Accretive Operators in Linear 2-Normed Spaces

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Abstract

In this paper we discuss some properties of resolvents of an accretive operator in linear 2-normed spaces, focusing on the concept of contraction mapping and the unique fixed point of contraction mappings in linear 2- normed spaces. Also, we establish the existence of solution of strong accretive operator equation in linear 2-normed spaces.

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1 Introduction

The concept of 2- metric spaces, linear 2- normed spaces and 2-inner product spaces , introduced by S Gahler in 1963 , paved the way for a number of authors like, A White, Y J Cho, R Freese, C R Diminnie, to do work on possible applications of Metric geometry, Functional Analysis and Topology as a new tool. A systematic presentation of the recent results related to the Geometry of linear 2-normed spaces as well as an extensive list of the related references can be found in the book [1]. In [4] S Gahler introduced the following definition of linear 2-normed spaces. The study of accretive operators in various spaces has been done by the authors T Kato[6], Shih-sen Chang, Yeol Je Cho, Shin Min Kang [5]..

2 Preliminaries

Definition 2.1 (4) *Let X be a real linear space of dimension greater than 1 and $\|.,.\|$ be a real valued function on $X \times X$ satisfying the properties,*
A1:$\|x,y\| = 0$ *iff* x *and* y *are linearly dependent*
A2:$\|x,y\| = \|y,x\|$
A3:$\|\alpha x,y\| = |\alpha|\|y,x\|$
A4:$\|x+y,z\| \leq \|x,z\| + \|y,z\|$
then the function $\|.,.\|$ is called a 2-norm on X. The pair (X,$\|.,.\|$) is called a linear 2- normed space.

Some of the basic properties of 2-norms, they are non-negative and $\|x,y+\alpha x\| = \|x,y\|$ ∀ x,y ∈ X and $\alpha \in$ R.

The most standard example for a linear 2 - normed space is X = R^2 equipped with the following 2-norm,

$$\|x_1,x_2\| = \text{abs det}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \text{ where } x_i = (x_{i1},x_{i2}) \text{for i=1,2}$$

Every linear 2-normed space is a locally convex TVS. In fact, for a fixed b∈ X, P_b(x)= $\|x,b\|$ for x ∈ X is a semi norm and the family $\{P_b; b \in X\}$ of semi norms generates a locally convex topology on X.

Definition 2.2 (1) *Let $(X,\|.,.\|)$ be a linear 2- normed space, E be a subset of X then the closure of E is $\overline{E} = \{x \in X;$ there is a sequence x_n of E such that $x_n \to x$. We say, E is sequentially closed if $E = \overline{E}$.*

Definition 2.3 (1) *Let $(X,\|.,.\|)$ be a linear 2- normed space, then the mapping $T : X \to X$ is said to be sequentially continuous if $x_n \to x$ implies $Tx_n \to Tx$.*

Definition 2.4 (3) *An operator $T : D(T) \subset X \to X$ is said to be accretive if for every $z \in D(T)$*

$$\|x-y,z\| \leq \|(x-y)+\lambda(Tx-Ty),z\| \textit{ for all } x,y \in D(T) \textit{ and } \lambda > 0.$$

Definition 2.5 (3) *An operator $T : D(T) \subset X \to X$ is said to be m-accretive if $R(I + \lambda T) = X$ for $\lambda > 0$.*

Remark 2.6 (4) *Every closed subset of a 2-Banach space is complete.*

3 Main Results

Consider an accretive operator $T : D(T) \subset X \to X$ then for $\lambda > 0$,$(I+\lambda T)^{-1}$ exists.We have,$T : D(T) \to R(T)$ then $(I + \lambda T)$ is onto. Let$x_1, x_2 \in D(T)$ with $(I + \lambda T)x_1 = (I + \lambda T)x_2$ implies $(x_1 - x_2) + \lambda(Tx_1 - Tx_2) = 0$ implies $\|(x_1 - x_2) + \lambda(Tx_1 - Tx_2), z\| = 0$ for every $z \in X$. Since T is accretive , we have $\|x_1 - x_2, z\| \leq 0$ for every $z \in X$ implies $\|x_1 - x_2, z\| = 0$ for every $z \in X$ implies $x_1 - x_2 = 0$ implies $x_1 = x_2$. So $(I + \lambda T)$ is one-one.
Hence for $\lambda > 0$, $(I + \lambda T)^{-1}$exist.

Definition 3.1 *Let $(X, \|.,.\|)$ be a linear 2- normed space then an operator T on X is said to be non expansive if for each x,y ∈D(T), $\|Tx - Ty, z\| \leq \|x - y, z\|$ for every z∈X. T is said to be expansive if for each x,y ∈D(T), $\|Tx - Ty, z\| > \|x - y, z\|$ for every z∈X.*

Proposition 3.2 *.Let $(X, \|.,.\|)$ be a linear 2- normed space and $T : D(T) \subset X \to X$ be an operator. If $(I + \lambda T)$ is expansive for all $\lambda > 0$ then T is accretive.*

Proof: The proof is immediate from the definition 3.1.

Remark 3.3 *If T is an accretive operator on a 2-normed space X then $(I + \lambda T)^{-1}$ is non expansive.*

Definition 3.4 *Let $(X, \|.,.\|)$ be a linear 2- normed space E be a non empty subset of X and $e \in E$ then E is said to be e-bounded if there exists some $M > 0$ such that $\|x, e\| \leq M$ for all $x \in E$. If for all $e \in E$, E is e-bounded then E is called a bounded set.*

Let $(X, \|.,.\|)$be a linear 2- normed space and T be an m-accretive operator on X. For n=1,2,3.., Define the resolvent of T as, $J_n(x) = (I+n^{-1}T)^{-1}(x)$ and the Yosida approximation $T_n(x) = n(I - J_n)(x)$ for all $x \in X$ and $\lambda > 0$.

Proposition 3.5 *.Let $(X, \|.,.\|)$ be a linear 2- normed space and T be a m-operator then*

(i)$\|J_nx - J_ny, z\| \leq \|(x - y), z\|$

(ii)$\|T_nx - T_ny, z\| \leq 2n\|(x - y), z\|$ *for all* $x, y \in X$

Proof:

(i) Since T is accretive in X,we have $(I+\alpha T)^{-1}$ is non expansive implies for each x,y ∈D(T), $\|(I+\alpha T)^{-1}x-(I+\alpha T)^{-1}y,z\| \leq \|x-y,z\|$ for every z∈X

Take $\alpha = 1/n$ then $\alpha > 0$. So $\|J_nx - J_ny, z\| \leq \|(x-y), z\|$.

(ii) We have for each x,y ∈D(T),

$$\begin{aligned}\|T_nx - T_ny, z\| &= \|n(I-J_n)x-(I-J_n)y, z\| \\ &= \|n(x-y)+n(J_nx-J_ny), z\| \\ &\leq n\|x-y,z\| + n\|J_nx - J_ny, z\| \\ &\leq n\|x-y,z\| + n\|x-y,z\| \\ &= 2n\|x-y,z\|\end{aligned}$$

for all $z \in X$

Proposition 3.6 *.Let $(X, \|.,.\|)$ be a linear 2- normed space and E be a non empty bounded subset of X and $T : E \to E$ be an accretive operator then there exists some $M > 0$ and for $x \in E$, $\|T_nx, z\| \leq M$ for all $z \in E$.*

Proof: Let $x \in E$ then

$$T_nx = n(x - J_nx) = n(J_n(J_n^{-1}x) - J_nx) = n(J_n(I + n^{-1}T)x - J_nx)$$

So, by Proposition 3.5 (i);

$$\begin{aligned}\|T_nx, z\| = \|n(J_n(I+n^{-1}T)x - J_nx), z\| &\leq n\|(I+n^{-1}T)x - x, z\| \\ &= \|Tx, z\| \leq M\end{aligned}$$

for all $z \in E$

Proposition 3.7 *.Let $(X, \|.,.\|)$ be a linear 2- normed space and E be a non empty bounded subset of X and $T : E \to E$ be an accretive operator . If $x \in \overline{E}$ then $J_nx \longrightarrow x$.*

Proof: Let $x \in E$ then$\|x - J_nx, z\| = n^{-1}\|T_nx, z\| \leq n^{-1}M$ for some $M > 0$ and for all $z \in E$.

ie; $\|x - J_nx, z\| \longrightarrow 0$ as $n \longrightarrow \infty$ implies $J_nx \longrightarrow x$

If x $\in \overline{E}$ then there exists a sequence $\{x_m\} \in E$ such that $x_m \longrightarrow x$. Since $x_m \in E$ we have $J_n(x_m) \longrightarrow x_m$.

We have, for all $z \in E$

$$\begin{aligned}\|J_nx - x, z\| &= \|J_nx - J_n(x_m) + J_n(x_m) - x_m + x_m - x, z\| \\ &\leq \|J_nx - J_n(x_m), z\| + \|J_n(x_m) - x_m, z\| + \|x_m - x, z\| \\ &\leq \|x - x_m, z\| + \|J_n(x_m) - x_m, z\| + \|x_m - x, z\| \longrightarrow 0 \text{ as}\end{aligned}$$

$n \longrightarrow \infty$

Hence $J_nx \longrightarrow x$.

Definition 3.8 *Let $(X, \|.,.\|)$ be a linear 2- normed space then the mapping $T : X \longrightarrow X$ is said to be a contraction if there exists some $k \in (0,1)$ such that $\|Tx - Ty, z\| \leq k\|x - y, z\|$ for all x,y,z $\in$ X.*

Lemma 3.9 *Let $(X, \|.,.\|)$ be a linear 2- normed space then every contraction $T : X \to X$ is sequentially continuous.*

Proof: Since T is a contraction, there exists some $k \in (0,1)$ such that $\|Tx - Ty, z\| \leq k\|x - y, z\|$ for all x,y,z $\in$ X.

Let $\{x_n\}$ be a sequence in X such that $x_n \longrightarrow x$.

Then, $\|Tx_n - Tx, z\| \leq k\|x_n - x, z\| \longrightarrow 0$ as $n \longrightarrow \infty$ implies $Tx_n \longrightarrow Tx$. So T is sequentially continuous.

Next lemma proves the analogues of Banach fixed point theorem in Linear 2-normed spaces.

Lemma 3.10 *Let $(X, \|.,.\|)$ be a linear 2- normed space and E be a non empty closed and bounded subset of X. Let $T : E \to E$ be a contraction then T has a unique fixed point on X.*

Proof: Since T is a contraction, there exists some $k \in (0,1)$ such that

$$\|Tx - Ty, z\| \leq k\|x - y, z\| \text{ for all x,y,z} \in \text{X}.$$

We have, $\|T^2x - T^2y, z\| = \|T(Tx) - T(Ty), z\| \leq k\|Tx - Ty, z\| \leq k^2\|x - y, z\|$ for all $z \in E$. Similarly $\|T^nx - T^ny, z\| \leq k^n\|x - y, z\|$ for all $z \in E$.

Let $x_0 \in E$ then construct a sequence depending on x_0.

Let $x_1 = Tx_0, x_2 = Tx_1, x_3 = Tx_2, ..., x_n = Tx_{n-1}$ then $x_1 = Tx_0, x_2 = T^2x_0, x_3 = T^3x_0, ..., x_n = T^nx_0$.

We show that $\{x_n\}$ is a cauchy sequence in E. Let $m, n > 0$ with $m > n$. Take $m = n + p$ then for any $z \in E$,

$\|x_n - x_m, z\| = \|x_n - x_{n+p}, z\|$

$= \|(x_n - x_{n+1}) + (x_{n+1} - x_{n+2}) + x_{n+2} - ... + (x_{n+p-1} - x_{n+p}), z\|$

$\leq \|x_n - x_{n+1}, z\| + \|x_{n+1} - x_{n+2}, z\| + ... + \|x_{n+p-1} - x_{n+p}, z\|$

$= \|T^nx_0 - T^nx_1, z\| + \|T^{n+1}x_0 - T^{n+1}x_1, z\| + ... + \|T^{n+p-1}x_0 - T^{n+p-1}x_1, z\|$

$\leq k^n\|x_0 - x_1, z\| + k^{n+1}\|x_0 - x_1, z\| + ... + k^{n+p-1}\|x_0 - x_1, z\|$

$\leq k^n\|x_0 - x_1, z\|(1 + k + k^2 + ...) = (\frac{k^n}{1-k})\|x_0 - x_1, z\|$

1e; $\|x_n - x_m, z\| \leq (\frac{k^n}{1-k})\|x_0 - x_1, z\|$ for all $z \in E$

Since E is bounded, there exists $M > 0$ such that $\|x_0 - x_1, z\| \leq M$ for all $z \in E$

So, $\|x_n - x_m, z\| \leq (\frac{k^n M}{1-k})$ for all $z \in E$ implies $\|x_n - x_m, z\| \longrightarrow 0$ as $n \longrightarrow \infty$, since $k \in (0,1)$ imples $\{x_n\}$ is a cauchy sequence in E, So $\{x_n\}$ converges to some x in E.

Since T is sequentially continuous, $Tx = \lim Tx_n = \lim x_{n+1} = x$ as $n \longrightarrow \infty$. Therefore T has a fixed point in E.

Now, we have to prove that such a fixed point is unique.

Let $y \in E$ with $y \neq x$ and $Ty \neq y$.

Assume that $Ty = y$ then $\|x - y, z\| = \|Tx - Ty, z\| \leq k\|x - y, z\|$ implies $k \geq 1$, a contradiction to $k \in (0,1)$. So, $Ty \neq y$. Hence the fixed point of T is unique in E.

Definition 3.11 *Let $(X, \|.,.\|)$ be a linear 2- normed space then the mapping $T : X \to X$ is said to be strong accretive if for every $z \in X$*

$$\|(\lambda - k)(x - y), z\| \leq \|(\lambda - 1)(x - y) + (Tx - Ty), z\|$$

for all $x, y \in X$ and $\lambda > k$, and $k \in (0,1)$.

Theorem 3.12 *Let $(X, \|.,.\|)$ be a linear 2- normed space and E be a non empty closed and bounded subset of X. Let $T : E \to X$ be a strong accretive mapping. If $(I + T)(E) \supset E$ then the equation $Tx = \theta$ has a solution in E, where θ is the zero vector in X.*

Proof: Since T is strong accretive implies

$$\|(\lambda - k)(x - y), z\| \leq \|(\lambda - 1)(x - y) + (Tx - Ty), z\|$$

for all $x, y \in X$ and $\lambda > k$, and $k \in (0,1)$

Put $\lambda = 2$ then $\|(2 - k)(x - y), z\| \leq \|(x - y) + (Tx - Ty), z\|$

implies $|2 - k|\|(x - y), z\| \leq \|(I + T)(x) - (I + T)(y)), z\|$

implies $\|(x - y), z\| \leq \frac{\|(I+T)(x)-(I+T)(y),z\|}{|2-k|}$

Since $(I + T)(E) \supset E$ we get,

$x, y \in D[(I + T)^{-1}]$ implies $x = (I + T)^{-1}(u)$ and $y = (I + T)^{-1}(v)$ for some $u, v \in E$

So, $\|(I + T)^{-1}(u) - (I + T)^{-1}(v), z\| \leq \frac{\|u-v,z\|}{|2-k|}$

ie; $(I + T)^{-1} : E \to E$ is a contraction mapping. By the Lemma 3.10 $(I + T)^{-1}$ has a fixed point x_0 in E. ie; $x_0 = (I + T)^{-1}(x_0)$ implies $x_0 = (I + T)(x_0)$ implies $Tx_0 = \theta$.

Hence,the equation $Tx = \theta$ has a solution in E.

Corollary 3.13 *Let $(X, \|.,.\|)$ be a linear 2- normed space and E be a non empty closed and bounded subset of X. Let $T : E \to E$ be a strong accretive mapping. If $(I + T)(E) = E$ then $R(T) = E$.*

Proof:

For any $p \in E$. Take $T_0 = T - p$

Since T is strong accretive we have,

$\|(\lambda - k)(x - y), z\| \leq \|(\lambda - 1)(x - y) + (Tx - Ty), z\|$ for all $x, y \in X$ and $\lambda > k$, and $k \in (0, 1)$

We have, T_0 is also strong accretive.

Therefore, by theorem 3.12, there exists $x_0 \in E$ such that $T_0(x_0) = \theta$ implies $Tx_0 - p = \theta$ implies $Tx_0 = p$ for every $p \in E$. Therefore, $R(T) = E$.

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