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LINEARLY INDEPENDENT ELEMENTS IN N-GROUPS WITH FINITE GOLDIE DIMENSION

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Abstract. The concepts *linearly independent elements* and *u-linearly independent elements* in an N-group G where N is a near-ring, were introduced and studied. A few important results in the theory of vector spaces were generalized to N-groups.

0. Introduction

Throughout, by a near-ring, we mean a zero-symmetric right near-ring. N stands for a near-ring and G stands for an N-group. $\langle X \rangle$ denotes the ideal generated by X for a given subset X of G and $\langle a \rangle$ denotes $\langle \{a\} \rangle$.

The concept of finite Goldie dimension in N-groups was introduced by Reddy and Satyanarayana[4]. An ideal H of G is said to have *finite Goldie dimension* (FGD) if H does not contain an infinite number of non-zero ideals of G whose sum is direct. An ideal A of G is said to be *essential* in an ideal B of G (denote as, $A \leq_e B$) if I is an ideal of G contained in B and $A \cap I = (0)$ imply $I = (0)$. An ideal A of G is said to be *uniform* if every non-zero ideal I of G, which is contained in A, is essential in A.

In [4], the authors proved that if an ideal H of G has FGD, then there exist finite number of uniform ideals $U_i, 1 \leq i \leq k$ of G whose sum is direct and essential in H. This number k is independent of choice of U_i's and k, is called the *Goldie dimension* of H. In this case, we write $k = \dim H$.

For preliminary definitions and results we refer [3, 4, 5, 7].

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Definition 0.1. (Satyanarayana[6]):

(i) An ideal K of G is said to be N-*simple* if K contains no non-zero proper N-subgroups;

(ii) an ideal H of G is said to be finite N-*completely reducible* if H can be written as a sum of finite number of N-simple ideals of G;

(iii) an ideal K of G is said to be *strictly maximal* if G/K is N-simple.

The intersection of all strictly maximal ideals of G is denoted by $J(G)$.

Note 0.2. If I is an ideal of G, then I is N-simple $\Rightarrow$ I is simple $\Rightarrow$ I is uniform.

Now a straightforward verification provides the following results.

Result 0.3. (a) Let U be an ideal of G. Then the following are equivalent:

(i) U is uniform, and

(ii) $0 \neq x \in U$ and $0 \neq y \in U \Rightarrow \langle x\rangle \cap \langle y\rangle \neq (0)$.

(b) Suppose $f : G \to G^1$ is an isomorphism and $I_i, 1 \leq i \leq n$, are ideals of G. Then

(i) the sum of ideals $I_i, 1 \leq i \leq n$ of G is direct in G if and only if the sum of ideals $f(I_i), 1 \leq i \leq n$ of G^1 is direct in G^1; and

(ii) $I_1 \leq_e I_2$ if and only if $f(I_1) \leq_e f(I_2)$.

Now we prove a preliminary lemma, which will be used in later sections.

Lemma 0.4. *Let $f : G \to G^1$ be an epimorphism. Then for any $x \in G, f(\langle x\rangle) = \langle f(x)\rangle$.*

Proof. Following the notation 0.1 given in [4], we have that

$$\langle x\rangle = \bigcup_{i=0}^{\infty} A_i, \quad \text{where} \quad A_{i+1} = A_i^* \cup A_i^0 \cup A_i^+ \ \text{for all} \quad i \geq 0,$$

and

$$A_i^* = \{g + x - g \mid x \in A_i, g \in G\},$$
$$A_i^0 = \{a - b \mid a, b \in A_i\},$$
$$A_i^+ = \{n(g + a) - ng \mid a \in A_i, n \in N, g \in G\} \quad \text{with} \quad A_0 = \{x\}.$$

Also $\langle f(x)\rangle = \bigcup_{i=0}^{\infty} B_i$, where $B_{i+1} = B_i^* \cup B_i^0 \cup B_i^+$ with $B_0 = \{f(x)\}$.

We verify that $B_0 = f(A_0), \ldots, B_i = f(A_i)$ for all $i \geq 0$. Now $B_0 = \{f(x)\} = f(A_0)$. Suppose the induction hypothesis: $B_k = f(A_k)$. Now we have to verify that $B_{k+1} = f(A_{k+1})$.

Part (i): Take $y \in B_{k+1} = B_k^* \cup B_k^0 \cup B_k^+$

Suppose $y \in B_k^*$. Then $y = g + b - g$ for some $b \in B_k$ and $g \in G^1$. Now $b \in B_k = f(A_k) \Rightarrow b = f(a)$ for some $a \in A_k$. Since f is onto, there exists $g_1 \in G$ such that $f(g_1) = g$. Now $y = g + b - g = f(g_1) + f(a) - f(g_1) = f(g_1 + a - g_1) \in f(A_k^*) \subseteq f(A_{k+1})$. Therefore $B_k^* \subseteq f(A_{k+1})$. Similarly we can prove that $B_k^0 \subseteq f(A_{k+1})$ and $B_k^* \subseteq f(A_{k+1})$. Thus $B_{k+1} \subseteq f(A_{k+1})$.

Part (ii): Let $z \in A_k^*$. Then $z = g + a - g$ for some $a \in A_k, g \in G \Rightarrow f(z) = f(g + a - g) = f(g) + f(a) - f(g) \in B_k^*$ (since $f(a) \in f(A_k) = B_k$). Therefore $f(A_k^*) \subseteq B_k^*$. Similarly we can show that $f(A_k^0) \subseteq B_k^0, f(A_k^+) \subseteq B_k^+$.

From the parts (i) and (ii), we have $f(A_{k+1}) = B_{k+1}$.

By mathematical induction, we conclude that $f(A_i) = B_i$ for all $i = 1, 2, \ldots$. Hence

$$\langle f(x) \rangle = \bigcup_{i=0}^{\infty} B_i = \bigcup_{i=0}^{\infty} f(A_i) = f\left(\bigcup_{i=0}^{\infty} A_i\right) = f(\langle x \rangle). \qquad \square$$

Definitions 0.5. (i) A subset S of G is said to be *small* in G if $S + K = G$ and K is an ideal of G imply $K = G; G$ is said to be *hollow* if every proper ideal of G is small in G.

(ii) G is said to have *finite spanning dimension* (FSD) if for any decreasing sequence of N-subgroups $X_0 \supset X_1 \supset X_2 \ldots$ of G such that X_i is an ideal of X_{i-1}, there exists an integer k such that X_j is small in G for all $j \geq k$.

1. Linearly independent elements and spanning sets

Definition 1.1. Let X be a subset of G. X is said to be a *linearly independent*(l.i.) set if the sum $\sum_{a \in X} \langle a \rangle$ is direct. If $\{a_i \mid 1 \leq i \leq n\}$ is a l.i. set, then we say that the elements $a_i, 1 \leq i \leq n$ are *linearly independent*. If X is not a l.i. set, then we say that X is a *linearly dependent*(l.d.) set.

Definition 1.2. An element $0 \neq u \in G$ is said to be *uniform element* (*u-element*) if $\langle u \rangle$ is an uniform ideal of G.

The proof of the following remark is straightforward.

Remark 1.3. Suppose G has FGD. If H is a non-zero ideal of G, then H contains a u-element.

Result 1.4. (i) If $a_i, 1 \leq i \leq m$ are l.i. elements in G then $m \leq n$ where $n = \dim G$.

(ii) $\dim G$ is equal to the least upper bound of the set A where $A = \{m \mid \text{m is a positive integer such that } a_i \in G, 1 \leq i \leq m \text{ are } l.i.\}$

(iii) If $n = \dim G$ and $a_i, 1 \leq i \leq n$ are l.i., then each (a_i) is an uniform ideal (in other words, each a_i is a u-element).

Proof. (i) and (ii) follows from Corollary 2.5 and Theorem 2.4 of [4].

(iii) If (a_k) is not uniform for some $1 \leq k \leq n$, then (a_k) contains two non-zero ideals A and B such that $A \cap B = (0)$. By the Remark 1.3, there exist u-elements $u \in A$ and $v \in B$. Now $a_{1,}a_{2,\ldots,}a_{k-1,}u, v, a_{k+1,\ldots,}a_n$ are linearly independent, a contradiction. □

Definition 1.5. If $n = \dim G$ and $a_i, 1 \leq i \leq n$ are l.i., then $\{a_i \mid 1 \leq i \leq n\}$ is called an *essential basis* for G.

A straightforward verification gives the following note.

Note 1.6. (i) G has FGD $\Leftrightarrow$ l.i. subset X of G is a finite set.

(ii) Suppose that $\dim G = n$ and $X \subseteq G$. If X is a l.i. set, then we have: $|X| = n \Leftrightarrow X$ is a maximal l.i. set $\Leftrightarrow X$ is an essential basis for G.

Lemma 1.7. *Let $f : G \to G^1$ be an isomorphism and $x_i \in G$, $1 \leq i \leq k$. Then*

(i) *$x_1, x_2, \ldots, x_k$ are l.i. elements in $G \Leftrightarrow f(x_1), f(x_2), \ldots, f(x_k)$ are l.i. elements in G^1;*

(ii) *$u \in G$ is a u-element in $G \Leftrightarrow f(u)$ is a u-element in G^1.*

(iii) *$x_1, x_2, \ldots, x_k$ are u-l.i. elements in $G \Leftrightarrow f(x_1), f(x_2), \ldots, f(x_k)$ are u-l.i. elements in G^1.*

Proof. (i) Follows from Result 0.3 and Lemma 0.4 (ii) In a contrary way, suppose that $\langle f(u)\rangle$ is not uniform. Take $w_1, w_2 \in \langle f(u)\rangle$ such that $\langle w_1\rangle \cap \langle w_2\rangle = (0)$. By the Lemma 0.4, there exist $u_1, u_2 \in \langle u\rangle$ such that $w_1 = f(u_1), w_2 = f(u_2)$. Since w_1, w_2 are linearly independent, by Lemma 1.7, $u_{1,}u_2$ are linearly independent, which imply that u cannot be a u-element. The rest follows similarly. □

Now we generalize the concept essentially spanned given in [1] to N-groups.

Definition 1.8. Let H be an ideal of G and $X \subseteq H$. We say that H is

(i) essentially spanned by a collection of ideals $\{I_\alpha\}_{\alpha\in\Delta}$ of G (or $\{I_\alpha\}_{\alpha\in\Delta}$ spans H essentially) if $\sum_{\alpha\in\Delta} I_\alpha$ is essential in H;

(ii) spanned by a collection of ideals $\{I_\alpha\}_{\alpha\in\Delta}$ of G (or $\{I_\alpha\}_{\alpha\in\Delta}$ spans H) if $\sum_{\alpha\in\Delta} I_\alpha = H$;

(iii) essentially spanned by X (or X spans H essentially or X is an essentially spanning set for H) if $\sum_{x\in X}\langle x\rangle$ is essential in H;

(iv) spanned by X (or X spans H or X is a spanning set for H) if $\sum_{x\in X}\langle x\rangle = H$.

NOTE 1.9. (i) $\{I_\alpha\}_{\alpha\in\Delta}$ spans $H \Rightarrow \{I_\alpha\}_{\alpha\in\Delta}$ spans H essentially and the converse is not true;

(ii) X spans $H \Rightarrow X$ spans H essentially and the converse is not true.

EXAMPLES 1.10. Let $N = Z$, the near-ring of integers and $G = Z$, the additive group of integers. Now G is an N-group.

(i) Consider $I = 2Z$. Clearly the ideal I is essential in G. Therefore I spans G essentially. Since $I \neq G$, we have that I do not spans G.

(ii) Write $X = \{2\}$. Clearly $\sum_{x\in X}\langle x\rangle = 2Z = I$ is essential in G. So X spans G essentially. Since $\sum_{x\in X}\langle x\rangle \neq G$, we have that X do not spans G.

DEFINITION 1.11. Let H be an ideal of G. (a) H is said to be

(i) *finitely spanned ideal* if it has a finite spanning set;

(ii) H is said to be *finitely essentially spanned* ideal if it has a finite essential spanning set;

(b) If $X = \{x\}$ and X essentially spans H, then H is called *essentially cyclic ideal*.

NOTE 1.12. If U is a uniform ideal, then U is an essentially cyclic ideal. Every essentially cyclic ideal need not be uniform.

For example, write $N = Z$, the near-ring of integers; and $G = Z_6$ the group of integers modulo 6. Now G is an N-group. Since $G = \langle 1\rangle$, G is essentially cyclic N-group, which is not uniform.

RESULT 1.13. Suppose G is semi-simple N-group with FGD. Then

(i) there exist simple ideals $H_1, H_2, \ldots, H_n$ such that $H_1 \oplus H_2 \oplus \cdots \oplus H_n = G$; and

(ii) there exist uniform ideals $U_i, 1 \leq i \leq n$ such that $G = U_1 \oplus U_2 \oplus \cdots \oplus U_n$.

Proof. In a contrary way, suppose that G cannot be expressed as a sum of finite number of simple ideals. Let H_1 be a simple ideal. Clearly $H_1 \neq G$. Then there exists a simple ideal H_2 such that $H_1 \neq H_2$. Now $H_1 \cap H_2 = (0)$ and so $H_1 + H_2$ is a direct sum. Since $H_1 + H_2 \neq G$, there exists a simple ideal H_3 of G such that $H_1 + H_2 + H_3 \neq H_1 + H_2$.

If $H_3 \cap (H_1 + H_2) \neq (0)$, then $H_3 \subseteq H_1 + H_2$ (since H_3 is simple) $\Rightarrow$ $H_1+H_2+H_3 = H_1+H_2$, a contradiction. Therefore $H_3 \cap (H_1+H_2) = (0)$ and so the sum $H_1 + H_2 + H_3$ is direct. Now $H_1 + H_2 + H_3 \neq G$. If we continue this process up to infinite steps, we get an infinite chain $H_1 \subset H_1 \oplus H_2 \subset H_1 \oplus H_2 \oplus H_3 \subset \cdots$ such that for each m, $H_1 \oplus H_2 \oplus \cdots \oplus H_m$ is not essential in $H_1 \oplus H_2 \cdots \oplus H_m \oplus H_{m+1}$, a contradiction, since G has FGD. Hence there exists n such that $G = H_1 \oplus H_2 \oplus \cdots \oplus H_n$.

(ii) Follows form (i) and Note 0.2 □

2. u-linearly independent elements

DEFINITIONS 2.1. A subset X of G is said to be *u-linearly independent*(u-l.i.) set if every element of X is a u-element and X is a l.i. set. Elements $a_i \in G, 1 \leq i \leq n$ are said to be u-l.i. if $\{a_i \mid 1 \leq i \leq n\}$ is a u-l.i. set. A u-l.i. set X is said to be a *maximal* u-l.i. set if $X \cup \{b\}$ is a u-linearly dependent set for each uniform element $0 \neq b \in G \backslash X$.

RESULT 2.2. Suppose $n = \dim G$ and $a_i, 1 \leq i \leq n$ are l.i. elements. Then

(i) $a_i, 1 \leq i \leq n$ are u-l.i. elements;
(ii) $\{a_i \mid 1 \leq i \leq n\}$ forms an essential basis for G; and
(iii) the conditions (i) and (ii) are equivalent.

Proof.

(i) Follows from Result 1.4 (iii);
(ii) Follows from (i); and
(iii) Clear. □

RESULT 2.3. Suppose G has FGD. Then

(i) If $b_i, 1 \leq i \leq k$ are l.i. elements then there exist u-elements $a_i \in \langle b_i \rangle, 1 \leq i \leq k$ such that $a_i, 1 \leq i \leq k$ are u-l.i. elements;
(ii) If H is a non-zero ideal of G then there exists a u-l.i. set $X = \{a_i | \ 1 \leq i \leq k\}$ such that $\langle X \rangle = \bigoplus_{i=1}^{k} \langle a_i \rangle \leq_e H$.

Moreover $\dim H = k$.

Proof. (i) Follows from Remark 1.3.
(ii) Clear. □

THEOREM 2.4. (i) If G has FSD, then there exist u-l.i elements $u_i, 1 \leq i \leq m$ in $G/J(G)$ which spans $G/J(G)$. Moreover $G/J(G)$ can be written as a direct sum of finite number of uniform ideals;

(ii) If G has FSD, then $G/J(G)$ has FGD.

Proof. By the Lemma 1.3 of Satyanarayana[7], $G/J(G)$ is finite N-completely reducible. This means there exist N-simple ideals $K_1, K_2, \ldots, K_m$ of $G/J(G)$ such that $G/J(G) = K_1 \oplus K_2 \oplus \cdots \oplus K_m$. Let $0 \neq u_i \in K_i, 1 \leq i \leq m$. Now by Note 0.2, each u_i is a u-element. Since $(0) \neq \langle u_i \rangle \subseteq K_i$ and K_i is simple, we have that $\langle u_i \rangle = K_i$ for $1 \leq i \leq m$. So $G/J(G) = \langle u_1 \rangle \oplus \langle u_2 \rangle \oplus \cdots \oplus \langle u_m \rangle$.

(ii) Follows from (i). □

Theorem 2.5. *Suppose G has FGD. Then K is a complement ideal of G if and only if there exist u-l.i. elements $u_1 + K, u_2 + K, \ldots, u_m + K$ in G/K which spans G/K essentially with $m = \dim G - \dim K$.*

Proof. Suppose that K is a complement ideal of G. Since K is a complement, by Result 1.6 of [7], $\dim(G/K) = \dim G - \dim K$. So $\dim(G/K) = m$. Hence by the Theorem 2.4 of [4], G/K contains m uniform ideals whose sum is direct and essential in G/K. We select one and only one non-zero element from each of these uniform ideals. Suppose these elements are $u_i + K, 1 \leq i \leq m$. Now $u_i + K, 1 \leq i \leq m$ are l.i. and $\langle u_1 + K \rangle \oplus \cdots \oplus \langle u_m + K \rangle$ is essential in G/K.

Conversely suppose that there exist u-l.i elements $u_1 + K, \ldots, u_m + K$ in G/K which spans G/K essentially. Then $\langle u_1 + K \rangle \oplus \cdots \oplus \langle u_m + K \rangle \leq_e G/K$. This shows that $\dim(G/K) = m$. Therefore $\dim(G/K) = m = \dim G - \dim K$. Now by Result 1.6 of [7], K is complement ideal of G. □

Theorem 2.6. *Suppose G has FGD and $\dim G = n$, $k < n$. If $u_1, u_2, \ldots, u_k$ are u-l.i elements of G, then there exist $u_{k+1}, \ldots, u_n$ in G such that $u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n$ span G essentially.*

Proof. Given that $u_i, 1 \leq i \leq k$ are u-l.i. elements. Write $H = \langle u_1 \rangle \oplus \cdots \oplus \langle u_k \rangle$. Now $\dim H = k$. Since $\dim H = k < n = \dim G$, by Corollary 2.5 of [4], we have that H is not essential in G. Since H is not essential in G, there exists a non-zero ideal H^1 of G such that $H \cap H^1 = (0)$. By Zorn's Lemma, $B = \{I |\ I$ is a non-zero ideal of G such that $H \cap I = (0)\}$ contains a maximal element, say J. By Result 1.4 of [6], $H \oplus J$ is an essential ideal in G. Now $n = \dim G = \dim(H \oplus J) = \dim H + \dim J = k + \dim J$. This implies that $\dim J = n - k$. Since $\dim J = n - k$, there exist u-l.i. elements $v_1, v_2, \ldots, v_{n-k}$ in J such that the sum of $\langle v_i \rangle, 1 \leq i \leq n - k$ is direct and essential in J. Since $H \cap J = (0)$, by Corollary 2.3 of [4] we have that $\langle u_1 \rangle \oplus \cdots \oplus \langle u_k \rangle \oplus \langle v_1 \rangle \oplus \cdots \oplus \langle v_{n-k} \rangle$ is essential in G. This shows that $u_1, u_2, \ldots, u_k, v_1, \ldots, v_{n-k}$ are u-l.i. elements which span G essentially. □

Theorem 2.7. *If G has FGD, then the following are equivalent:*

(i) $\dim G = n$;

(ii) *There exist n uniform ideals $U_i, 1 \leq i \leq n$, whose sum is direct and essential in G;*
(iii) *The maximum number of u-l.i. elements in G is n;*
(iv) *n is maximum with respect to the property that for any given $\{x_1, x_2, \ldots, x_k\}$ of u-l.i. elements with $k < n$, there exist $x_{k+1}, \ldots,$ x_n such that $\{x_1, x_2, \ldots, x_n\}$ are u-l.i. elements;*
(v) *The maximum number of l.i. elements that can span G essentially is n;*
(vi) *The minimum number of u-l.i. elements that can span G essentially is n.*

Proof. (i) $\Leftrightarrow$ (ii): Follows from 2.4 of [4].
(i) $\Rightarrow$ (iii): Follows from the Result 1.4.
(iii) $\Rightarrow$ (ii): Is a routine verification.
(i) $\Rightarrow$ (iv): Follows from the Theorem 2.5 and the Result 1.4.
(iv) $\Rightarrow$ (iii): Clear.
(i) $\Leftrightarrow$ (v): Follows from the Result 1.4.
(i) $\Rightarrow$ (vi): In a contrary way, suppose that there exist u-l.i. elements $u_i, 1 \leq i \leq k$, and $k < n$ such that $u_i, 1 \leq i \leq k$ span G essentially. This means $\langle u_1 \rangle \oplus \cdots \oplus \langle u_n \rangle \leq_e G$. By the Theorem 2.5, there exist u-l.i elements $u_{k+1}, \ldots, u_n$ such that $u_1, u_2, \ldots, u_n$ are u-l.i. elements. This implies $(\langle u_1 \rangle \oplus \cdots \oplus \langle u_k \rangle) \cap (\langle u_{k+1} \rangle) = (0)$.

Since $\langle u_1 \rangle \oplus \cdots \oplus \langle u_k \rangle \leq_e G$, we have that $\langle u_{k+1} \rangle = (0) \Rightarrow u_{k+1} = 0$, a contradiction.
(vi) $\Rightarrow$ (ii): Clear. □

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