LINEARLY INDEPENDENT ELEMENTS IN
N-GROUPS WITH FINITE GOLDIE DIMENSION

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Abstract. The concepts linearly independent elements and u-linearly independent elements in an N-group G where N is a near-ring, were introduced and studied. A few important results in the theory of vector spaces were generalized to N-groups.

0. Introduction

Throughout, by a near-ring, we mean a zero-symmetric right nearring. N stands for a near-ring and G stands for an N-group. \( \langle X \rangle \) denotes the ideal generated by X for a given subset X of G and \( \langle a \rangle \) denotes \( \langle \{ a \} \rangle \).

The concept of finite Goldie dimension in N-groups was introduced by Reddy and Satyanarayana[4]. An ideal \( H \) of G is said to have finite Goldie dimension (FGD) if \( H \) does not contain an infinite number of non-zero ideals of G whose sum is direct. An ideal \( A \) of G is said to be essential in an ideal \( B \) of G (denote as, \( A \leq_e B \)) if I is an ideal of G contained in \( B \) and \( A \cap I = (0) \) imply \( I = (0) \). An ideal \( A \) of G is said to be uniform if every non-zero ideal I of G, which is contained in \( A \), is essential in \( A \).

In [4], the authors proved that if an ideal \( H \) of G has FGD, then there exist finite number of uniform ideals \( U_i, 1 \leq i \leq k \) of G whose sum is direct and essential in \( H \). This number \( k \) is independent of choice of \( U_i \)'s and \( k \), is called the Goldie dimension of \( H \). In this case, we write \( k = \dim H \).

For preliminary definitions and results we refer [3, 4, 5, 7].

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Definition 0.1. (Satyanarayana[6]):
(i) An ideal $K$ of $G$ is said to be $N$-simple if $K$ contains no non-zero proper $N$-subgroups;
(ii) an ideal $H$ of $G$ is said to be finite $N$-completely reducible if $H$ can be written as a sum of finite number of $N$-simple ideals of $G$;
(iii) an ideal $K$ of $G$ is said to be strictly maximal if $G/K$ is $N$-simple.
The intersection of all strictly maximal ideals of $G$ is denoted by $J(G)$.

Note 0.2. If $I$ is an ideal of $G$, then $I$ is $N$-simple $\Rightarrow I$ is simple $\Rightarrow I$ is uniform.

Now a straightforward verification provides the following results.

Result 0.3.
(a) Let $U$ be an ideal of $G$. Then the following are equivalent:
(i) $U$ is uniform, and
(ii) $0 \neq x \in U$ and $0 \neq y \in U \Rightarrow \langle x \rangle \cap \langle y \rangle \neq (0)$.
(b) Suppose $f : G \to G^1$ is an isomorphism and $I_i, 1 \leq i \leq n$, are ideals of $G$. Then
(i) the sum of ideals $I_i, 1 \leq i \leq n$ of $G$ is direct in $G$ if and only if the sum of ideals $f(I_i), 1 \leq i \leq n$ of $G^1$ is direct in $G^1$; and
(ii) $I_1 \leq e I_2$ if and only if $f(I_1) \leq e f(I_2)$.

Now we prove a preliminary lemma, which will be used in later sections.

Lemma 0.4. Let $f : G \to G^1$ be an epimorphism. Then for any $x \in G, f(\langle x \rangle) = \langle f(x) \rangle$.

Proof. Following the notation 0.1 given in [4], we have that
$$\langle x \rangle = \bigcup_{i=0}^{\infty} A_i, \quad \text{where} \quad A_i+1 = A_i^* \cup A_i^0 \cup A_i^+, \quad \text{for all} \quad i \geq 0,$$
and
$$A_i^* = \{ g + x - g \mid x \in A_i, g \in G \},$$
$$A_i^0 = \{ a - b \mid a, b \in A_i \},$$
$$A_i^+ = \{ n(g + a) - ng \mid a \in A_i, n \in N, g \in G \} \quad \text{with} \quad A_0 = \{ x \}.$$ 
Also $\langle f(x) \rangle = \bigcup_{i=0}^{\infty} B_i$, where $B_{i+1} = B_i^* \cup B_i^0 \cup B_i^+$ with $B_0 = \{ f(x) \}$.

We verify that $B_0 = f(A_0), \ldots, B_3 = f(A_4)$ for all $i \geq 0$. Now $B_0 = \{ f(x) \} = f(A_0)$. Suppose the induction hypothesis: $B_k = f(A_k)$. Now we have to verify that $B_{k+1} = f(A_{k+1})$.

Part (i): Take $y \in B_{k+1} = B_k^* \cup B_k^0 \cup B_k^+$
Suppose \( y \in B^+_k \). Then \( y = g + b - g \) for some \( b \in B_k \) and \( g \in G^1 \).

Now \( b \in B_k = f(A_k) \Rightarrow b = f(a) \) for some \( a \in A_k \). Since \( f \) is onto, there exists \( g_1 \in G \) such that \( f(g_1) = g \). Now \( y = g + b - g = f(g_1) + f(a) - f(g_1) = f(g_1 + a - g_1) \in f(A_k) \subseteq f(A_{k+1}) \). Therefore \( B_k^+ \subseteq f(A_{k+1}) \).

Similarly we can prove that \( B_k^0 \subseteq f(A_{k+1}) \) and \( B_k^* \subseteq f(A_{k+1}) \). Thus \( B_{k+1}^* \subseteq f(A_{k+1}) \).

**Part (ii):** Let \( z \in A_k^* \). Then \( z = g + a - g \) for some \( a \in A_k, g \in G \Rightarrow f(z) = f(g + a - g) = f(g) + f(a) - f(g) \in B_k^+ \) (since \( f(a) \in f(A_k) = B_k \)). Therefore \( f(A_k^*) \subseteq B_k^+ \). Similarly we can show that \( f(A_k^0) \subseteq B_k^0, f(A_k^+) \subseteq B_k^+ \).

From the parts (i) and (ii), we have \( f(A_{k+1}) = B_{k+1} \).

By mathematical induction, we conclude that \( f(A_i) = B_i \) for all \( i = 1, 2, \ldots \). Hence

\[
\langle f(x) \rangle = \bigcup_{i=0}^{\infty} B_i = \bigcup_{i=0}^{\infty} f(A_i) = f \left( \bigcup_{i=0}^{\infty} A_i \right) = f(\langle x \rangle). \quad \square
\]

**Definitions 0.5.**

(i) A subset \( S \) of \( G \) is said to be **small** in \( G \) if \( S + K = G \) and \( K \) is an ideal of \( G \) imply \( K = G \); \( G \) is said to be **hollow** if every proper ideal of \( G \) is small in \( G \).

(ii) \( G \) is said to have **finite spanning dimension** (FSD) if for any decreasing sequence of \( N \)-subgroups \( X_0 \supset X_1 \supset X_2 \ldots \) of \( G \) such that \( X_j \) is an ideal of \( X_{j-1} \), there exists an integer \( k \) such that \( X_j \) is small in \( G \) for all \( j \geq k \).

1. **Linearly independent elements and spanning sets**

**Definition 1.1.** Let \( X \) be a subset of \( G \). \( X \) is said to be a **linearly independent** (l.i.) set if the sum \( \sum_{a \in X} \langle u \rangle \) is direct. If \( \{a_i | 1 \leq i \leq n \} \) is a l.i. set, then we say that the elements \( a_i, 1 \leq i \leq n \) are **linearly independent**. If \( X \) is not a l.i. set, then we say that \( X \) is a **linearly dependent** (l.d.) set.

**Definition 1.2.** An element \( 0 \neq u \in G \) is said to be **uniform element** (u-element) if \( \langle u \rangle \) is an uniform ideal of \( G \).

The proof of the following remark is straightforward.

**Remark 1.3.** Suppose \( G \) has FGD. If \( H \) is a non-zero ideal of \( G \), then \( H \) contains a u-element.
RESULT 1.4. (i) If \( a_i, 1 \leq i \leq m \) are l.i. elements in \( G \) then \( m \leq n \) where \( n = \dim G \).

(ii) \( \dim G \) is equal to the least upper bound of the set \( A \) where \( A = \{ m \mid m \text{ is a positive integer such that } a_i \in G, 1 \leq i \leq m \text{ are l.i.} \} \)

(iii) If \( n = \dim G \) and \( a_i, 1 \leq i \leq n \) are l.i., then each \((a_i)\) is an
uniform ideal (in other words, each \( a_i \) is a \( u \)-element).

Proof. (i) and (ii) follows from Corollary 2.5 and Theorem 2.4 of [4].

(iii) If \((a_k)\) is not uniform for some \( 1 \leq k \leq n \), then \((a_k)\) contains two
non-zero ideals \( A \) and \( B \) such that \( A \cap B = (0) \). By the Remark 1.3, there
exist \( u \)-elements \( u \in A \) and \( v \in B \). Now \( a_1, a_2, \ldots, a_{k-1}, u, v, a_{k+1}, \ldots, a_n \) are
linearly independent, a contradiction.

Definition 1.5. If \( n = \dim G \) and \( a_i, 1 \leq i \leq n \) are l.i., then
\[ \{ a_i \mid 1 \leq i \leq n \} \]
is called an essential basis for \( G \).

A straightforward verification gives the following note.

Note 1.6. (i) \( G \) has FGD \( \Leftrightarrow \) l.i. subset \( X \) of \( G \) is a finite set.

(ii) Suppose that \( \dim G = n \) and \( X \subseteq G \). If \( X \) is a l.i. set, then we
have: \( |X| = n \Leftrightarrow X \) is a maximal l.i. set \( \Leftrightarrow X \) is an essential basis for \( G \).

Lemma 1.7. Let \( f : G \to G^1 \) be an isomorphism and \( x_i \in G, 1 \leq i \leq k \).
Then

(i) \( x_1, x_2, \ldots, x_k \) are l.i. elements in \( G \) \( \Leftrightarrow \) \( f(x_1), f(x_2), \ldots, f(x_k) \) are
l.i. elements in \( G^1 \);

(ii) \( u \in G \) is a \( u \)-element in \( G \) \( \Leftrightarrow \) \( f(u) \) is a \( u \)-element in \( G^1 \).

(iii) \( x_1, x_2, \ldots, x_k \) are \( u \)-l.i. elements in \( G \) \( \Leftrightarrow \) \( f(x_1), f(x_2), \ldots, f(x_k) \) are
\( u \)-l.i. elements in \( G^1 \).

Proof. (i) Follows from Result 0.3 and Lemma 0.4 (ii) In a contrary
way, suppose that \( f(u) \) is not uniform. Take \( w_1, w_2 \in \langle f(u) \rangle \) such that
\( \langle w_1 \rangle \cap \langle w_2 \rangle = (0) \). By the Lemma 0.4, there exist \( u_1, u_2 \in \langle u \rangle \) such
that \( w_1 = f(u_1), w_2 = f(u_2) \). Since \( w_1, w_2 \) are linearly independent, by
Lemma 1.7, \( u_1, u_2 \) are linearly independent, which imply that \( u \) cannot
be a \( u \)-element. The rest follows similarly.

Now we generalize the concept essentially spanned given in [1] to
\( N \)-groups.

Definition 1.8. Let \( H \) be an ideal of \( G \) and \( X \subseteq H \). We say that
\( H \) is

(i) essentially spanned by a collection of ideals \( \{ I_\alpha \}_{\alpha \in \Delta} \) of \( G \) (or
\( \{ I_\alpha \}_{\alpha \in \Delta} \) spans \( H \) essentially) if \( \sum_{\alpha \in \Delta} I_\alpha \) is essential in \( H \);
(ii) spanned by a collection of ideals \( \{ I_\alpha \}_{\alpha \in \Delta} \) of \( G \) (or \( \{ I_\alpha \}_{\alpha \in \Delta} \) spans \( H \)) if \( \sum_{\alpha \in \Delta} I_\alpha = H \);

(iii) essentially spanned by \( X \) (or \( X \) spans \( H \) essentially or \( X \) is an essentially spanning set for \( H \)) if \( \sum_{x \in X} \langle x \rangle \) is essential in \( H \);

(iv) spanned by \( X \) (or \( X \) spans \( H \) or \( X \) is a spanning set for \( H \)) if \( \sum_{x \in X} \langle x \rangle = H \).

Note 1.9. (i) \( \{ I_\alpha \}_{\alpha \in \Delta} \) spans \( H \) \( \Rightarrow \) \( \{ I_\alpha \}_{\alpha \in \Delta} \) spans \( H \) essentially and the converse is not true;

(ii) \( X \) spans \( H \) \( \Rightarrow \) \( X \) spans \( H \) essentially and the converse is not true.

Examples 1.10. Let \( N = Z \), the near-ring of integers and \( G = Z \), the additive group of integers. Now \( G \) is an \( N \)-group.

(i) Consider \( I = 2Z \). Clearly the ideal \( I \) is essential in \( G \). Therefore \( I \) spans \( G \) essentially. Since \( I \neq G \), we have that \( I \) do not spans \( G \).

(ii) Write \( X = \{ 2 \} \). Clearly \( \sum_{x \in X} \langle x \rangle = 2Z = I \) is essential in \( G \). So \( X \) spans \( G \) essentially. Since \( \sum_{x \in X} \langle x \rangle \neq G \), we have that \( X \) do not spans \( G \).

Definition 1.11. Let \( H \) be an ideal of \( G \). (a) \( H \) is said to be

(i) finitely spanned ideal if it has a finite spanning set;

(ii) \( H \) is said to be finitely essentially spanned ideal if it has a finite essential spanning set;

(b) If \( X = \{ x \} \) and \( X \) essentially spans \( H \), then \( H \) is called essentially cyclic ideal.

Note 1.12. If \( U \) is a uniform ideal, then \( U \) is an essentially cyclic ideal. Every essentially cyclic ideal need not be uniform.

For example, write \( N = Z \), the near-ring of integers; and \( G = Z_6 \) the group of integers modulo 6. Now \( G \) is an \( N \)-group. Since \( G = \langle 1 \rangle \), \( G \) is essentially cyclic \( N \)-group, which is not uniform.

Result 1.13. Suppose \( G \) is semi-simple \( N \)-group with FGD. Then

(i) there exist simple ideals \( H_1, H_2, \ldots, H_n \) such that \( H_1 \oplus H_2 \oplus \cdots \oplus H_n = G \); and

(ii) there exist uniform ideals \( U_i, 1 \leq i \leq n \) such that \( G = U_1 \oplus U_2 \oplus \cdots \oplus U_n \).

Proof. In a contrary way, suppose that \( G \) cannot be expressed as a sum of finite number of simple ideals. Let \( H_1 \) be a simple ideal. Clearly \( H_1 \neq G \). Then there exists a simple ideal \( H_2 \) such that \( H_1 \neq H_2 \). Now \( H_1 \cap H_2 = (0) \) and so \( H_1 + H_2 \) is a direct sum. Since \( H_1 + H_2 \neq G \), there exists a simple ideal \( H_3 \) of \( G \) such that \( H_1 + H_2 + H_3 \neq H_1 + H_2 \).
If \( H_3 \cap (H_1 + H_2) \neq (0) \), then \( H_3 \subseteq H_1 + H_2 \) (since \( H_3 \) is simple) \( \Rightarrow \) \( H_1 + H_2 + H_3 = H_1 + H_2 \), a contradiction. Therefore \( H_3 \cap (H_1 + H_2) = (0) \) and so the sum \( H_1 + H_2 + H_3 \) is direct. Now \( H_1 + H_2 + H_3 \neq G \). If we continue this process up to infinite steps, we get an infinite chain \( H_1 \subset H_1 \oplus H_2 \subset H_1 \oplus H_2 \oplus H_3 \subset \cdots \) such that for each \( m \), \( H_1 \oplus H_2 \oplus \cdots \oplus H_m \) is not essential in \( H_1 \oplus H_2 \oplus \cdots \oplus H_m \oplus H_{m+1} \), a contradiction, since \( G \) has FGD. Hence there exists \( n \) such that \( G = H_1 \oplus H_2 \oplus \cdots \oplus H_n \).

(ii) Follows from (i) and Note 0.2

\[\]

2. \textbf{u-linearly independent elements}

DEFINITIONS 2.1. A subset \( X \) of \( G \) is said to be \textit{u-linearly independent} (u-l.i.) set if every element of \( X \) is a u-element and \( X \) is a l.i. set. Elements \( a_i \in G, 1 \leq i \leq n \) are said to be \textit{u-l.i.} if \( \{a_i \mid 1 \leq i \leq n\} \) is a u-l.i. set. A u-l.i. set \( X \) is said to be a \textit{maximal} u-l.i. set if \( X \cup \{b\} \) is a u-linearly dependent set for each uniform element \( 0 \neq b \in G \setminus X \).

RESULT 2.2. Suppose \( n = \dim G \) and \( a_i, 1 \leq i \leq n \) are l.i. elements. Then

\begin{enumerate}
  \item \( a_i, 1 \leq i \leq n \) are u-l.i. elements;
  \item \( \{a_i \mid 1 \leq i \leq n\} \) forms an essential basis for \( G \); and
  \item the conditions (i) and (ii) are equivalent.
\end{enumerate}

\textbf{Proof.} \protect

(i) Follows from Result 1.4 (iii);
(ii) Follows from (i); and
(iii) Clear.

RESULT 2.3. Suppose \( G \) has FGD. Then

\begin{enumerate}
  \item If \( b_i, 1 \leq i \leq k \) are l.i. elements then there exist u-elements \( a_i \in \langle b_i \rangle, 1 \leq i \leq k \) such that \( a_i, 1 \leq i \leq k \) are u-l.i. elements;
  \item If \( H \) is a non-zero ideal of \( G \) then there exists a u-l.i. set \( X = \{a_i \mid 1 \leq i \leq k\} \) such that \( (X) = \bigoplus_{i=1}^{k} \langle a_i \rangle \leq H \).
\end{enumerate}

Moreover \( \dim H = k \).

\textbf{Proof.} (i) Follows from Remark 1.3.
(ii) Clear.

THEOREM 2.4. \protect

(i) If \( G \) has FSD, then there exist u-l.i elements \( u_i, 1 \leq i \leq m \) in \( G/J(G) \) which spans \( G/J(G) \). Moreover \( G/J(G) \) can be written as a direct sum of finite number of uniform ideals;
(ii) If \( G \) has FSD, then \( G/J(G) \) has FGD.
Proof. By the Lemma 1.3 of Satyanarayana[7], $G/J(G)$ is finite $N$-completely reducible. This means there exist $N$-simple ideals $K_1, K_2, \ldots, K_m$ of $G/J(G)$ such that $G/J(G) = K_1 \oplus K_2 \oplus \cdots \oplus K_m$. Let $0 \neq u_i \in K_i, 1 \leq i \leq m$. Now by Note 0.2, each $u_i$ is a $u$-element. Since $(0) \neq \langle u_i \rangle \subseteq K_i$ and $K_i$ is simple, we have that $\langle u_i \rangle = K_i$ for $1 \leq i \leq m$. So $G/J(G) = \langle u_1 \rangle \oplus \langle u_2 \rangle \oplus \cdots \oplus \langle u_m \rangle$.

(ii) Follows from (i). \hfill \Box

Theorem 2.5. Suppose $G$ has FGD. Then $K$ is a complement ideal of $G$ if and only if there exist $u$-l.i. elements $u_1 + K, u_2 + K, \ldots, u_m + K$ in $G/K$ which spans $G/K$ essentially with $m = \dim G - \dim K$.

Proof. Suppose that $K$ is a complement ideal of $G$. Since $K$ is a complement, by Result 1.6 of [7], $\dim(G/K) = \dim G - \dim K$. So $\dim(G^i/K) = m$. Hence by the Theorem 2.4 of [4], $G/K$ contains $m$ uniform ideals whose sum is direct and essential in $G/K$. We select one and only one non-zero element from each of these uniform ideals. Suppose these elements are $u_i + K, 1 \leq i \leq m$. Now $u_i + K, 1 \leq i \leq m$ are l.i. and $\langle u_1 + K \rangle \oplus \cdots \oplus \langle u_m + K \rangle$ is essential in $G/K$.

Conversely suppose that there exist $u$-l.i elements $u_1 + K, \ldots, u_m + K$ in $G/K$ which spans $G/K$ essentially. Then $\langle u_1 + K \rangle \oplus \cdots \oplus \langle u_m + K \rangle \leq e G/K$. This shows that $\dim(G/K) = m$. Therefore $\dim(G/K) = m = \dim G - \dim K$. Now by Result 1.6 of [7], $K$ is complement ideal of $G$. \hfill \Box

Theorem 2.6. Suppose $G$ has FGD and $\dim G = n, k < n$. If $u_1, u_2, \ldots, u_k$ are $u$-l.i elements of $G$, then there exist $u_{k+1}, \ldots, u_n$ in $G$ such that $u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n$ span $G$ essentially.

Proof. Given that $u_i, 1 \leq i \leq k$ are $u$-l.i. elements. Write $H = \langle u_1 \rangle \oplus \cdots \oplus \langle u_k \rangle$. Now $\dim H = k$. Since $\dim H = k < n = \dim G$, by Corollary 2.5 of [4], we have that $H$ is not essential in $G$. Since $H$ is not essential in $G$, there exists a non-zero ideal $H^1$ of $G$ such that $H \cap H^1 = (0)$. By Zorn’s Lemma, $B = \{ I \mid I$ is a non-zero ideal of $G$ such that $H \cap I = (0) \}$ contains a maximal element, say $J$. By Result 1.4 of [6], $H \ominus J$ is an essential ideal in $G$. Now $n = \dim G = \dim(H \ominus J) = \dim H + \dim J = k + \dim J$. This implies that $\dim J = n - k$. Since $\dim J = n - k$, there exist $u$-l.i. elements $v_1, v_2, \ldots, v_{n-k}$ in $J$ such that the sum of $\langle v_i \rangle, 1 \leq i \leq n - k$ is direct and essential in $J$. Since $H \cap J = (0)$, by Corollary 2.3 of [4] we have that $\langle u_1 \rangle \oplus \cdots \oplus \langle u_k \rangle \oplus \langle v_1 \rangle \oplus \cdots \oplus \langle v_{n-k} \rangle$ is essential in $G$. This shows that $u_1, u_2, \ldots, u_k, v_1, \ldots, v_{n-k}$ are $u$-l.i. elements which span $G$ essentially. \hfill \Box

Theorem 2.7. If $G$ has FGD, then the following are equivalent:

(i) $\dim G = n$;
(ii) There exist \( n \) uniform ideals \( U_i, 1 \leq i \leq n \), whose sum is direct and essential in \( G \);

(iii) The maximum number of u-l.i. elements in \( G \) is \( n \);

(iv) \( n \) is maximum with respect to the property that for any given \{\( x_1, x_2, \ldots, x_k \)\} of u-l.i. elements with \( k < n \), there exist \( x_{k+1}, \ldots, x_n \) such that \{\( x_1, x_2, \ldots, x_n \)\} are u-l.i. elements;

(v) The maximum number of l.i. elements that can span \( G \) essentially is \( n \);

(vi) The minimum number of u-l.i. elements that can span \( G \) essentially is \( n \).

Proof. (i) \( \Leftrightarrow \) (ii): Follows from 2.4 of [4].

(i) \( \Rightarrow \) (iii): Follows from the Result 1.4.

(iii) \( \Rightarrow \) (iv): Is a routine verification.

(i) \( \Rightarrow \) (iv): Follows from the Theorem 2.5 and the Result 1.4.

(iv) \( \Rightarrow \) (iii): Clear.

(i) \( \Leftrightarrow \) (v): Follows from the Result 1.4.

(i) \( \Rightarrow \) (vi): In a contrary way, suppose that there exist u-l.i. elements \( u_i, 1 \leq i \leq k \), and \( k < n \) such that \( u_i, 1 \leq i \leq k \) span \( G \) essentially. This means \( \langle u_1 \rangle \oplus \cdots \oplus \langle u_n \rangle \leq e G \). By the Theorem 2.5, there exist u-l.i elements \( u_{k+1}, \ldots, u_n \) such that \( u_1, u_2, \ldots, u_n \) are u-l.i. elements. This implies \( \langle u_1 \rangle \oplus \cdots \oplus \langle u_k \rangle \cap \langle u_{k+1} \rangle = (0) \).

Since \( \langle u_1 \rangle \oplus \cdots \oplus \langle u_k \rangle \leq e G \), we have that \( \langle u_{k+1} \rangle = (0) \Rightarrow u_{k+1} = 0 \), a contradiction.

(vi) \( \Rightarrow \) (ii): Clear.

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