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Strong (Weak) Edge-Edge Domination Number of a Graph

R. S. Bhat

Department of Mathematics
Manipal Institute Of Technology
Manipal, India, 576 104
rs.bhat@manipal.edu

S. S. Kamath

Department of Mathematics
National Institute of Technology Karnataka
Surathkal, India, 574 025

Surekha R. Bhat

Department of Mathematics
Milagres College
Kallianpur, Udupi, India, 576 103

Abstract

For any edge $\boldsymbol{x = uv}$ of an isolate free graph $\boldsymbol{G(V,E)}, \langle \boldsymbol{N[x]} \rangle$ is the subgraph induced by the vertices adjacent to u and v in G. We say that an edge x, e-dominates an edge y if $\boldsymbol{y} \in \langle \boldsymbol{N[x]} \rangle$. A set $\boldsymbol{L \subseteq E}$ is an Edge-Edge Dominating Set (EED-set) if every edge in $\boldsymbol{E - L}$ is e-dominated by an edge in L. The *edge-edge domination number* $\boldsymbol{\gamma_{ee}(G)}$ is the cardinality of a minimum EED-set. We find the relation ship between the new parameter and some known graph parameters.

Keywords: Edge-Edge Dominating sets (EED sets), Strong Edge-Edge Dominating sets (SEED sets)

1 Introduction

For any undefined terminologies refer [3]. Degree of an edge $\boldsymbol{x = uv}$, deg (*x*) is the number of edges adjacent to the edge *x*. Equivalently deg(*x*) = deg (*u*) + deg (*v*) - 2. Unless specified otherwise by a graph we mean a simple undirected isolate free and isolate edge free graph. For any edge $\boldsymbol{x = uv, N(x) = \{y \in E | y \text{ is adjacent to } x\}}$ and $\boldsymbol{N[x] = N(x) \cup \{x\}}$. And $\boldsymbol{\langle N[x] \rangle}$ is the sub graph induced by $\boldsymbol{N(u) \cup N(v)}$.

The strong weak domination was first introduced by Sampathkumar and Pushpalatha [12]. For any two adjacent vertices u and v in a graph $G(V, E)$, u strongly (weakly) dominates v if deg (*u*) ≥ deg(*v*) (deg (*u*) ≤ deg (*v*)). A set $D \subseteq V$ is a *dominating set* (*strong dominating set* [SD-set], *weak dominating set* [WD-set] respectively) of G if every $v \in V - D$ is dominated (strongly dominated, weakly dominated respectively) by some $u \in D$. The *domination number* $\gamma(G)$ (*strong domination number* $\gamma_s(G)$, *weak domination number* $\gamma_w(G)$ respectively) is the minimum cardinality of a dominating set (SD- set, WD-set respectively) of *G*. Similarly, this concept is extended to coverings , independent sets and matchings by S.S.Kamath and R.S.Bhat [2, 4 and 5]. Sampathkumar and P.S.Neeralagi [10, 11] defined the neighbourhood sets and line neighbourhood sets as follows. A set $S \subseteq V$ is a *neighbourhood set* (*n- set*) if $G = \bigcup_{v \in S} \langle N[v] \rangle$. A set $L \subseteq E$ is a *Line neighbourhood set* (*ln- set*) if $G = \bigcup_{x \in L} \langle N[x] \rangle$. The *neighbourhood number* $n_0 = n_0(G)$ [*line neighbourhood number* $n'_0 = n'_0(G)$] is the cardinality of a minimum n-set [ln-set] of *G*. Mixed domination was introduced in 1985 by R.Laskar and Ken Peters [8] and then in 1992 by, Sampathkumar and S.S.Kamath [9]. An edge *x*, *m- dominates* a vertex v if $v \in N[x]$. A set $L \subseteq E$ is an *Edge Vertex Dominating set* (EVD-set) if every vertex in G is *m-dominated* by an edge in *L*. The *edge vertex domination number* $\gamma_{ev}(G)$ is the minimum cardinality of an EVD-set. Strong (weak) Edge vertex domination studied by R.S.Bhat et.al [1] and Vertex Edge domination is studied by S.S. Kamath and R.S.Bhat [6].

2. Strong /Weak Edge Edge Dominating sets

Let $x, y \in E$, of an isolate free graph $G(V, E)$ then the edge *x*, *e*-dominates an edge y if $y \in \langle N[x] \rangle$. An edge *x* strongly (weakly) *e*-dominates an edge y if $y \in \langle N[x] \rangle$ and deg (*x*) ≥ deg (*y*) (deg (*x*) ≤ deg(*y*)).

A set $L \subseteq E$ is an *Edge-Edge Dominating set* (EED-set) if every edge in *E* - *L* is *e*-dominated by an edge in *L*. The *edge-edge domination number* $\gamma_{ee}(G)$ is the minimum cardinality of an EED-set. A set $L \subseteq E$ is a *Strong Edge-Edge Dominating set* (SEED-set) [*Weak Edge-Edge Dominating set* (WEED-set)] if every edge in $E - L$ is strongly (weakly) *e*-dominated by an edge in *L*. The *strong* (*weak*) *edge-edge domination number* $\gamma_{see}(G)$ $(\gamma_{wee}(G))$ is the minimum cardinality of a SEED-set (WEED-set).

We observe that the definition of EED set is a restatement of the definition of line neighbourhood set and hence we have $\gamma_{ee} = n_0'$. In [9] it is proved that $\gamma_{ev} \leq \gamma_{ee}$. Therefore we have $\gamma_{ev} \leq \gamma_{ee} = n_0'$. Since every SEED set and WEED set is an EED set we have $\gamma_{ee} \leq \gamma_{see}$ and $\gamma_{ee} \leq \gamma_{wee}$.

A set $L \subset E$ is said to be *Full Edge-Edge dominating* set (FEED set) if every edge in L is e-dominated by an edge in $E - L$. A set $L \subset E$ is said to be a *Full Strong Edge-Edge Dominating set* (FSEED-set) [*Full Weak Edge-Edge Dominating set* (FWEED-set)] if every edge in L, is weakly (strongly) e-dominated by an edge in $E - L$. The *FEED number* $f_{ee}(G)$ (*FSEED number* $f_{see}(G)$, *FWEED number* $f_{wee}(G)$ respectively) is the maximum cardinality of a FEED set (FSEED set, FWEED set respectively).

Example 1. Here, $\gamma_{ee}(G_1) = \gamma_{see}(G_1) = \gamma_{wee}(G_1) = 3$. The dotted edges in Fig.1a, Fig.1b represent the both γ_{ee}- set as well as γ_{see} -set and the dotted edges in Fig.1c is a γ_{wee} -set. More over $f_{ee}(G_1) = f_{wee}(G_1) = f_{see}(G_1) = 6$. The dark edges in each figure form f_{ee} -set, f_{wee} -set and f_{see} -set respectively. We also observe that $\gamma_{ev}(G_1) = 2 < 3 = \gamma_{ee}(G_1)$.

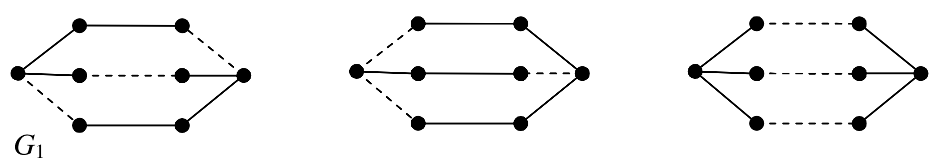

Fig 1.a Fig 1.b Fig 1.c

For any u, v in G the distance between u and $v, d(u,v)$ is the length of a shortest path between u and v. Let $, y \in E, a \in V$, and $x = uv$ then the distance between the vertex a and edge x, is defined as $d(x,a) = \min\{d(u,a), d(v,a)\}$ and the distance between the two edges $d(x,y) = \min\{d(y,u), d(y,v)\}$. This concept of distance between a vertex and an edge plays an important role in EED sets.

Remark 1. If L is a minimal EED set of an isolate edge free graph G, then E - L is also an EED set of G.

3. Main Results

Our first result gives a necessary and sufficient condition for a set $L \subseteq E$ to be an EED set of G in terms of distance between a vertex and an edge.

Proposition 1. *Let $G(V,E)$ be any graph without isolated edges. A set $L \subseteq E$ is an*

EED set of G if, and only if, for every edge $y = uv$ in $E - L$, there exists an edge $x \in L$ such that $d(x, u) \leq 1$ and $d(x, v) \leq 1$.

Proof. (⇒) Let L be an EED set of G. Then since G is a graph without isolated edges, every edge $y = uv$ in $E - L$ is e-dominated by some edge $x \in L$. Hence both $u, v \in N[x]$. This implies that both $d(x, u) \leq 1$ and $d(x, v) \leq 1$ hold as desired.

(⇐) Let $L \subseteq E$ and for every edge $y = uv$ in $E - L$, there exists an edge $x \in L$ such that $d(x, u) \leq 1$ and $d(x, v) \leq 1$. Suppose L is not an EED set of G, then there exists at least one edge $z = ab \in E - L$ such that z is not e-dominated by any edge in L. Then at least one of the conditions $d(x, a) \geq 2$ or $d(x, b) \geq 2$ holds, a contradiction to our assumption.

We shall next give a necessary and sufficient condition for minimality of an EED set.

Theorem 2.
(a) *A set L is a minimal EED set of G if, and only if, for any $x \in L$ one of the following two conditions holds.*
(i) *No edge in L e-dominates the edge x.* (ii)*There exists a $y \in E - L$ such that y is uniquely e -dominated by the edge x.*

(b) *A set L is a minimal SEED set* (*WEED set*) *of G if, and only if, for any $x \in L$ one of the following two conditions holds.*
(i) *No edge in L strongly* (*weakly*) *e-dominates the edge x.* (ii) *There exists an edge $y \in E - L$ which is uniquely strongly* (*weakly*) *e-dominated by the edge x.*

Proof. Assume L is a minimal EED set. Then for every $x \in L, L - \{x\}$ is not an EED set. This means that there exists a $y \in E - L$ such that y is not e-dominated by any edge in $L - \{x\}$. Then either $y = x$ *or* $y \in E - L$.

Case 1. If $y = x$: Then x is not e-dominated by any edge in L. Hence condition (i)holds. **Case 2.** If $y \in E - L$: Then y is not e-dominated by any edge in $L - \{x\}$ and y is e-dominated by L, together imply that y is uniquely e-dominated by the edge x. Hence condition (ii) holds. Conversely, suppose L is an EED set and for any $x \in L$ one of the following two conditions stated in the Proposition holds. We show that L is a minimal EED set. Then there exists a $x \in L$, such that $L - \{x\}$ is an EED set. This implies that x is e-dominated by an edge in L. That is x does not satisfy condition (i). Also if $L - \{x\}$ is an EED set, then every edge in $E - L$ is e-dominated by some edge in $L - \{x\}$. This implies that x does not uniquely e-dominate any edge in $E - L$. That is x does not satisfy condition (ii)-a contradiction to our assumption. Part (b) can be proved with the similar argument, hence we omit the proof.

Proposition 3. *Let $G(V, E)$ be any graph. For any set $L \subset E$,*
(i) *L is an EED set if, and only if, $E - L$ is a FEED set.*
(ii) *L is an SEED (WEED) set if, and only if, $E - L$ is a FWEED (FSEED) set.*

Proof. We prove (i) only. The proof of (ii) is similar. If L is an EED set then $E - L$ is a FEED set follows from Remark 2. Conversely if L is a FEED set then every edge in L is e-dominated by some edge in $E - L$. Clearly the edges in $E - L$ are e-dominated by them selves. Hence $E - L$ is an EED set.

Proposition 4. *Let $G\ (p, q)$ be any graph. Then*

$$\gamma_{ee} + f_{ee} = q \quad (1)$$

$$\gamma_{see} + f_{wee} = q \quad (2)$$

$$\gamma_{wee} + f_{see} = q \quad (3)$$

Proof. Let L be a minimum EED set of *G*. Then from Proposition 4, we have $E - L$ is a FEED set. Therefore $f_{ee} \geq |E - L| = q - \gamma_{ee}$ … (i). On the other hand if L is a maximum FEED set, again from Proposition 4, $E - L$ is an EED set of *G*. Hence $\gamma_{ee} \leq |E - L| = q - f_{ee}$ … (ii). Now (1) follows from (i) and (ii). Similarly the results (2) and (3) follow.

4. EE-degree, SEE-degree and WEE-degree

Several types of new degree are defined in [7]. The *Edge-Edge degree* (EE-degree) of an edge $x \in E$, $d_{ee}(x)$ is the number of edges e dominated by *x*. Equivalently $d_{ee}(x)$ is the number of edges in $N\langle[x]\rangle$. *Strong Edge-Edge degree* (SEE-degree) of an edge $x \in E$, $d_{see}(x)$ is the number of edges strongly e-dominated by *x*. Similarly WEE-deg (*x*) is $d_{wee}(x)$ defined. With respect to these degrees we get the following new graph parameters. Maximum EE-degree $\Delta_{ee}(x)$, minimum EE-degree $\delta_{ee}(x)$, Maximum SEE-degree $\Delta_{see}(x)$, minimum SEE-degree $\delta_{see}(x)$, Maximum WEE-degree $\Delta_{wee}(x)$, minimum WEE-degree $\delta_{wee}(x)$. An Edge x is called *SEE-Silent* (*WEE-Silent*), if $d_{see}(x) = 0$ ($d_{wee}(x) = 0$). A set $L \subset E$ is said to be *SEE-Silent set* (*WEE-Silent set*) if for every edge $x \in L$, $d_{see}(x) = 0$ ($d_{wee}(x) = 0$). The *SEE-Silent* (*WEE-Silent*) *number* $\eta_{see} = \eta_{see}(G)$ ($\eta_{wee} = \eta_{wee}(G)$) is the maximum cardinality of a SEE-silent set of G.

5. Bounds on γ_{ee}, γ_{see} and γ_{wee}

We now get some bounds in terms of Δ_{ee} and Δ_{wee}.

Proposition 5. *For any (p, q) graph G,*

$$\left\lceil\frac{q}{\Delta_{ee}}\right\rceil \leq \gamma_{ee} \leq \gamma_{see} \leq q - \eta_{see} \quad (4)$$

$$\left\lceil\frac{q}{\Delta_{wee}}\right\rceil \leq \gamma_{wee} \leq q - \eta_{wee} \quad (5)$$

Further the above bounds are sharp.

Proof. The lower bound in (4) is proved in [7]. Let $L \subseteq E$ be a η_{see} -set of G. Since every edge in L is a SEE-Silent, no edge in L strongly e-dominate any edge in G. Therefore $E - L$ is a SEED set of G. Hence $\gamma_{see} \leq |E_L| = q - \eta_{see}$. With similar argument we can prove the upper bound in (5). Since an edge in G can weakly e-dominate at most Δ_{wee} edges and it self, we need at least $\frac{q}{\Delta_{wee}}$ edges to weakly e-dominate all the edges. This implies the lower bound in (5). The above bounds are sharp as the upper bound in (4) is attained for P_4 and P_5 and the upper bound in (5) is attained for P_4.

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